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Selection through an associated characteristic

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I. REVIEW OF LITERATURE AND SCOPE OF THE PRESENT STUDY

A. Introduction

Let y_i $(i = 1, 2, \ldots, n)$ be observations on n objects or individuals. We take as best the object with the largest y-value. The following selection problems of increasing generality may be distinguished: (i) selecting one object which is best; (ii) selecting s objects which include the best object; and (iii) selecting s objects which include the $k(\leq s)$ best objects. There are many practical situations where it may be better to select more than one object in order to ensure that the selected subset contains the one (or more) best objects. For example, we may need to divide a number of, say n, candidates, such as students for a given test, into two groups - one that contains the k best candidates and one that contains mostly inferior candidates. Let the first group be of size $s(\geq k)$ and the second of size $n-s$. For given n and k, the subset size s may be determined so that the chosen subset contains the k best candidates with at least a given probability P.

In many experimental situations, the experimenter is also faced with a problem of selecting one or more out of n possible treatments. The treatments are usually characterized by the value of a parameter of interest θ and the experimenter is interested in choosing the treatments with the largest parameter values. The classical approach to this problem is to test the homogeneity null hypothesis $H_0: \theta_1 = \theta_2 = \ldots = \theta_n$, where θ_i are the values of the parameter, and if H_0 is rejected to follow up with multiple comparison methods. In the case of normal populations with means $\theta_1, \ldots, \theta_n$ and a common unknown variance σ^2, the test can be carried out using the F-ratio of the analysis of variance. This approach is inadequate and unrealistic in the sense that it is not formulated in a way to answer the experimenter's question, namely, how to identify the best treatment. For example, the method of least significant differences based on the t-test has been used to detect differences between the true unknown means of varieties and thereby to choose the population with the largest mean. But this does not provide an overall probability of correct selection. The same is true, in general, for methods based on multiple comparison techniques.

The formulation of the problem of selecting the best population as a multiple decision problem can be found in the early statistical literature including Paulson (1949), Bahadur (1950), and Bahadur and Robbins (1950). This formulation in the framework of selection and ranking procedures has been generally accomplished either using the "indifference zone approach" or the "subset selection approach". The first was proposed by Bechhofer (1954) and the second was introduced by Gupta (1956). Both approaches have a common target, the so-called "probability of correct selection (PCS)".

In order to explain each of these approaches briefly, let us consider the problem of selecting the population with the largest mean from n normal populations with unknown μ_i, $1 \leq i \leq n$, and a common known variance σ^2 . Let $\bar{X}_i$, $1 \leq i \leq n$, denote the means of n independent samples of size r from these populations and let $\mu_{[1]} \leq \mu_{[2]} \leq \cdots \leq \mu_{[n]}$ denote the ordered means. First, let us deal with the indifference zone approach. The natural procedure will be to select the population that yields the largest $\bar{X}_i$. The experimenter would like a guarantee that this procedure will pick the population having the largest μ_i with a probability not less than a specified value P . Since we do not know the true configuration of the μ_i , we look for the least favorable configuration (LFC) for which the PCS is minimized. This LFC is given by $\mu_1 = \mu_2 = \ldots = \mu_n$ and the corresponding PCS is $1/n$. Hence, the probability guarantee cannot be met however large the sample size r . A natural modification is to insist on the minimum probability guarantee whenever the best population is sufficiently superior to the next best. In other words, the experimenter specifies a positive constant δ and requires that PCS is at least P whenever $\mu_{[n]} - \mu_{[n-1]} \geq \delta$. Let $\Omega_\delta = \{\underline{\mu}: \mu_{[n]} - \mu_{[n-1]} \geq \delta\}$. The complement of Ω_δ is called the indifference zone (IZ). The LFC in Ω_δ is given by $\mu_{[1]} = \mu_{[2]} = \ldots = \mu_{[n-1]} = \mu_{[n]} - \delta$. The problem is then to determine the minimum sample size required in order to have PCS $\geq P$ for all $\underline{\mu}$ in Ω_δ .

In the subset selection approach, the experimenter wishes to choose a subset of the n populations so as to include the best population with at least a specified probability P . This procedure selects a random number of populations ranging from 1 to n , the actual number depending on the observations themselves. That is, this rule selects the population that yields $\bar{X}_i$ if and only if

$$\bar{X}_i \geq \max_{1\leq j\leq n} \bar{X}_j - \frac{d\sigma}{\sqrt{r}} ,$$

where $d = d(n,P) > 0$ is determined so that the PCS is at least P.

There is a large literature in this area. A few recent reviews in book or monograph form are Gibbons, Olkin, and Sobel (1977), Gupta and Panchapakesan (1979), and Gupta and Huang (1981). The methods and tables developed to date provide solutions to a variety of problems such as selecting the largest main effect (Lehmann (1961) and Gupta and Huang (1977)), selecting the treatment with the smallest variance (McDonald (1977)), and selecting the largest interaction in a two-factor experiment (Bechhofer, Santner, and Turnbull (1977)) in the fixed-effects model (Model I).

Consider now the situation when y_i is expensive to measure (e.g., destructive testing is needed) or represents a future observation (e.g. in a second decisive test). Then, selection may be based on associated observations x_i which are respectively inexpensive or available now. Using x_i, the same kinds of selection (i), (ii), and (iii) mentioned earlier can be considered.

The n pairs of observations (x_i, y_i) are assumed to be a random sample from a bivariate distribution with c.d.f. $F(x,y)$. Moreover, we may assume that high values of x tend to be accompanied by high values of y. The following questions arise immediately:

(a) If the object with the largest x-value is selected, what is the probability P_1 that it will have the largest y-value?

(b) Since P_1 will often be smaller than desired, we may wish to select a subset of objects with the highest x-values so as to have a sufficiently large probability P_2 that the best object is included in the chosen subset.

These questions can be answered with the help of theory developed in David, O'Connell, and Yang (1977). As in the selection problem (iii), of which (i) and (ii) are special cases, we can generalize the questions (a) and (b) as follows:

(c) Determine s so that the subset of objects with the s largest x-values contains the k largest y-values with sufficiently high probability.

Our approach to the solution of (c) is quite different from that used for (a) and (b). This new theory generalizes the theory of selection through an associated variable and enables us to apply it in many situations. As a special application, interesting and useful connections can be established with the components of variance model (Model II) whereas the large literature on the subject of selection mentioned before is confined to fixed-effects models.

B. Scope and Content of the Present Investigation

In the following Section C, the theory of the ranks of concomitants of order statistics is given since this forms the starting point for the evaluation of the probability of the generalized subset selection (c) of the last section. Accounts of the theory and applications of ranks of concomitants can be found in David (1973), O'Connell (1974), David and Galambos (1974), and David et al. (1977).

In Chapter II, the probability

$${}_n\Pi_{s:k} = \{\text{s objects with the largest } x_i \text{ include the } k \text{ largest } y_i\},$$

for an arbitrary absolutely continuous bivariate distribution, is derived and the properties of ${}_n\Pi_{s:k}$ are studied. In the important case of the bivariate normal distribution, we give the numerical values of ${}_n\Pi_{s:k}$ for various values of n, s, k, and ρ. Also, when $s = k = 1$, we give the value of ρ for selected values of $P = {}_n\Pi_{1:1}(\rho)$ as an inverse function. To satisfy ${}_n\Pi_{s:1}(\rho) \geq P$ for given values ρ, n, and P, the subset size s is tabulated. Computational problems are discussed since multiple integrals are involved and high numerical accuracy is desired.

In Chapter III, an approximate formula for ${}_n\Pi_{s:k}(\rho)$ when ρ is close to 1, is derived in the case of the bivariate normal distribution since the case $\rho \cong 1$ is of special interest in applications.

Chapter IV contains applications of the theory and related examples. The main application is to the components of variance model. Selection problems in the fixed-effects model have been studied during the last three decades but we cannot find any paper concerning selection of the "best" random effect in the components of variance model. We will show how the determination of the number of replications needed to achieve a specified probability of correct selection can be related to the results of Chapters II and III. Various kinds of random models or mixed models are studied.

C. Distribution of the Ranks of Concomitants of Order Statistics

Let (X_i, Y_i) $i = 1, 2, \ldots, n$ be n independent random variables having a common bivariate distribution corresponding to (X, Y). When the X_i are arranged in nondecreasing order as the order statistics $X_{r:n}$ $(r = 1, 2, \ldots, n)$, the Y-variate associated with $X_{r:n}$ may be denoted by $Y_{[r:n]}$ and termed the <u>concomitant of the r-th order statistic</u>. These concomitants have been studied extensively and have been put to a variety of uses, recent examples including David (1973), Gross (1973), Bhattacharyya (1974), David and Galambos (1974), O'Connell (1974), David *et al*. (1977), and Yang (1976, 1981). For convenience, the following notation concerning the distribution of random variables will be adopted:

$F(x,y)$ and $f(x,y)$ - joint c.d.f. and p.d.f. of random variables X and Y,

$F_X(x)$ and $f_X(x)$ - c.d.f. and p.d.f. of a random variable X,

$f(y|x) = f(x,y)/f_X(x)$ - conditional p.d.f. of Y given $X = x$.

Let $R_{r,n}$ denote the <u>rank</u> of $Y_{[r:n]}$ among the n Y_i. Let the indicator function $I(x)$ be defined by

$$I(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Then the rank of $Y_{[r,n]}$ is given by

$$R_{r,n} = \sum_{i=1}^{n} I(Y_{[r:n]} - Y_i) \quad r = 1, 2, \ldots, n \,. \tag{1.1}$$

Let X and Y have the absolutely continuous joint c.d.f. $F(x,y)$, with p.d.f. $f(x,y)$. Since $R_{r,n}$ is from (1.1) location and scale invariant with respect to both X and Y , we may take F and f to refer to the standardized variates. Writing $r(X_i)$ for the rank of X_i among the n X's , with a similar meaning for $r(Y_i)$, we have for $r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, n$,

$$\Pr\{R_{r,n} = s\} = \Pr\{r(Y_i) = s \text{ and } r(X_i) = r \text{ for some } i\} \,.$$

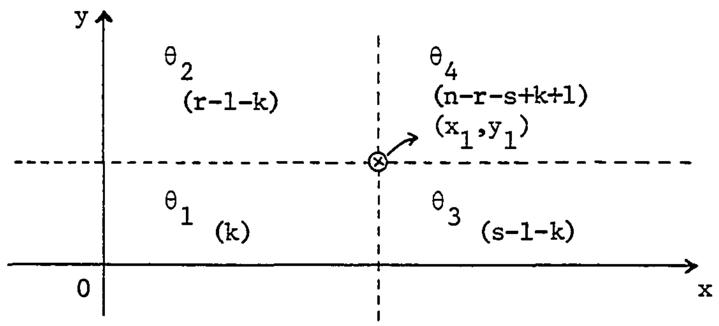
Since (X_i, Y_i) $i = 1, 2, \ldots, n$ are independent, identically distributed, we have

$$\begin{aligned} \Pr\{R_{r,n} = s\} &= \sum_{i=1}^{n} \Pr\{r(Y_i) = s \text{ and } r(X_i) = r\} \\ &= n \Pr\{r(Y_1) = s \text{ and } r(X_1) = r\} \,. \end{aligned} \tag{1.2}$$

The manner in which the compound event $\{r(Y_1) = s \text{ and } r(X_1) = r\}$ can occur is best seen from the following 2×2 frequency table with fixed marginals

	$Y_i < Y_1$	$Y_i > Y_1$	
$X_i < X_1$	θ_1 k	θ_2 $r-1-k$	$r-1$
$X_i > X_1$	θ_3 $s-1-k$	θ_4 $n-r-s+k+1$	$n-r$
	$s-1$	$n-s$	$n-1$

or from the following graph



where the four cell entries, θ_i $(i = 1, 2, 3, 4)$, are defined as follows:

$$\left.\begin{aligned}
\theta_1(x,y) &= \Pr\{X < x,\ Y < y\} = F(x,y)\\
\theta_2(x,y) &= \Pr\{X < x,\ Y > y\} = F_X(x) - F(x,y)\\
\theta_3(x,y) &= \Pr\{X > x,\ Y < y\} = F_Y(y) - F(x,y)\\
\theta_4(x,y) &= \Pr\{X > x,\ Y > y\} = 1 - F_X(x) - F_Y(y) + F(x,y)
\end{aligned}\right\} . \qquad (1.3)$$

By conditioning on X_1, Y_1, we then have from (1.2) and the frequency table

$$\Pr\{R_{r,n} = s\} = n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=\ell}^{u} C_k \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{n-r-s+k+1} f(x,y)\,dxdy\ , \qquad (1.4)$$

where

$$\ell = \max\ \{0,\ s+r-1-n\}\ ,$$

$$u = \min\ \{r-1,\ s-1\}\ ,$$

$$C_k(r,s,n) = \frac{(n-1)!}{k!(r-1-k)!(s-1-k)!(n-r-s+k+1)!}\ .$$

Equation (1.4) provides the discrete distribution of $R_{r,n}$.

From (1.2) the following two symmetry relations for $\Pi_{rs} = \Pr\{R_{r,n} = s\}$ may be obtained (David *et al.*, 1977).

Relation 1: If there exist monotone increasing transformations from X to X' and from Y to Y' such that the joint p.d.f. $g(x', y')$ of X' and Y' is symmetric (i.e., $g(x', y') = g(y', x')$), then

$$\Pi_{rs} = \Pi_{sr} \; ; \; r,s = 1, 2, \ldots, n \; . \tag{1.5}$$

Relation 2: If $f(x, y) = f(-x, -y)$, then

$$\Pi_{rs} = \Pi_{n-r+1,n-s+1} \; ; \; r,s = 1, 2, \ldots, n \; . \tag{1.6}$$

Note that we can find the moments of $R_{r,n}$ in O'Connell (1974).

II. PROBABILITY OF SELECTION THROUGH THE RANKS OF THE ASSOCIATED VARIABLE

A. Introduction

In Chapter I, we defined the concomitants and the ranks of concomitants of order statistics. Let (X_i,Y_i), $i = 1, \ldots, n$ be a paired random sample from some bivariate distribution with p.d.f. $f(x,y)$. For definiteness, we may assume any association to be such that high values of x tend to be accompanied by high values of y. If y is expensive to measure or represents a future observation, then selection may be based on associated observations x_i which are respectively inexpensive or available now. This situation is very common and has a great many examples in real life.

Several kinds of selection may be distinguished including (i) selection of the best object, (ii) subset selection: selection of s objects to include the best object, and (iii) generalized subset selection: selection of s objects to include the best $k(\leq s)$ objects. For the univariate case these selection problems have been studied widely. See Gibbons, Olkin, and Sobel (1977), Gupta and Panchapakesan (1979), and Gupta and Huang (1981). This chapter deals mainly with the derivation of the probability of selection (iii) of which the other two are special cases. Since the probability of correct selection will often be smaller than desired, we may wish to tabulate the probability for various subset sizes s in order to find sufficiently large subset size. Because of the progressive theoretical development in finding these selection probabilities, this chapter deals with the selection problems in the above order.

B. Selection of the Best Object

Let ${}_n\Pi_{s:k}$ be the probability that a subset of s objects with the s largest out of n x-values include the k largest y-values. In this section, we derive this probability when $s = k = 1$, i.e., ${}_n\Pi_{1:1}$. This can be done directly from the distribution of $R_{r,n}$, $r = 1, \ldots, n$. From (1.4), we have

$$
\begin{aligned}
{}_n\Pi_{1:1} &= \Pr\{R_{n,n} = n\} \\
&= n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-1}(x,y) f(x,y)\,dxdy \ . \qquad (2.1)
\end{aligned}
$$

When $f(x,y)$ is a bivariate normal p.d.f. with the correlation ρ, the numerical value of the probability ${}_n\Pi_{1:1}$ is given in Section E (Table 2.1) in this chapter for various n and ρ. Since ${}_n\Pi_{1:1}$ is a function of ρ, we can consider the inverse function to determine the correlation ρ for a given value P of ${}_n\Pi_{1:1}$. For selected values of P, the values of ρ are also tabulated (Table 2.2).

C. Subset Selection of the Best Object

Since the probability ${}_n\Pi_{1:1}$ will often be smaller than desired, we may wish to select a subset of objects with the largest x-values so as to have a sufficiently large probability P that the object with the largest y-value is included in the chosen subset. The size s of the subset $(s \leq n)$ now has to be determined for a given n, P, and $f(x,y)$. The probability ${}_n\Pi_{s:1}$ can be evaluated by accumulating the probabilities $\Pr\{R_{i,n} = n\}$, i.e.,

$$
\begin{aligned}
{}_n\Pi_{s:1} &= \sum_{i=n-s+1}^{n} \Pr\{R_{i,n} = n\} \\
&= \sum_{i=n-s+1}^{n} \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_1^{i-1}(x,y)\theta_3^{n-i}(x,y)f(x,y)\,dxdy \qquad (2.2)
\end{aligned}
$$

from equation (1.4). For fixed n , the probability ${}_n\Pi_{s:1}$ depends on s and the underlying p.d.f. $f(x,y)$. When $f(x,y)$ is symmetric (i.e., $f(x,y) = f(y,x)$), we have that from the relations (1.5) and (1.6)

$$
\begin{aligned}
{}_n\Pi_{s:1} &= \sum_{i=1}^{s} \Pr\{R_{n,n} = n-i+1\} \\
&= \sum_{i=1}^{s} \frac{n!}{(n-i)!(i-1)!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_1^{n-i}(x,y)\theta_2^{i-1}(x,y)f(x,y)\,dxdy . \qquad (2.3)
\end{aligned}
$$

To determine the size of subset s giving the desired probability P, we have to find the smallest value of s for which ${}_n\Pi_{s:1}$ exceeds P . For various values of n, s, and ρ in the bivariate normal case, ${}_n\Pi_{s:1}$ is tabulated in Section E (Table 2.3 with $k = 1$). Also, for given probability P , the minimum subset size s is tabulated (Table 2.4) as a function of ρ , i.e.,

$$s = \min_j\{{}_n\Pi_{j:1}(\rho) \geq P\} \quad . \qquad (2.4)$$

D. Generalized Subset Selection of the Best k Objects

1. Statement of the problem

As a more complex selection problem, we consider the following questions:

(i) What is the probability ${}_n\Pi_{s:k}$ that a subset of s objects with the largest x-values includes the largest k y-values?

(ii) What is the smallest value of subset size, s, ensuring with the probability at least P that objects with the largest k y-values are included among the s objects with the largest x-values?

To answer these questions, the theory of the previous two sections cannot be applied directly. We will derive the exact probability ${}_n\Pi_{s:k}$ and study its properties.

2. Derivation of the probability ${}_n\Pi_{s:k}$

Let (X_i, Y_i) $i = 1, \ldots, n$ be a set of independent paired random variables each having p.d.f. $f(x,y)$. In terms of the ranks of concomitants, the probability ${}_n\Pi_{s:k}$ is expressed by

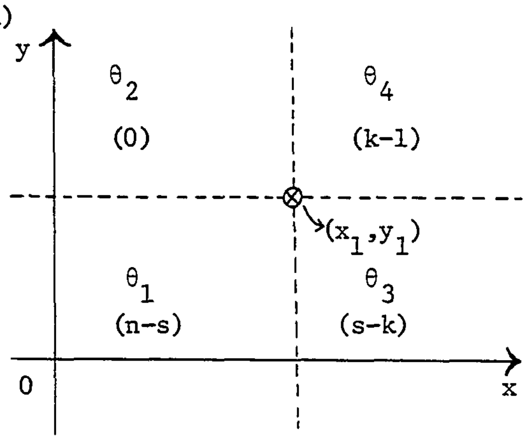
$${}_n\Pi_{s:k} = \Pr\{\{R_{n,n}, R_{n-1,n}, \ldots, R_{n-s+1,n}\} \supset \{n, n-1, \ldots, n-k+1\}\} .$$

The event on the right may be decomposed into three mutually exclusive parts to give

$$
\begin{aligned}
{}_n\Pi_{s:k} = {} & \Pr\{R_{n-s+1,n} = n-k+1 \text{ and } \{R_{n,n}, \ldots, R_{n-s+2,n}\} \supset \{n, \ldots, n-k+2\}\} \\
& + \Pr\{R_{n-s+1,n} \geq n-k+2 \text{ and } \{R_{n,n}, \ldots, R_{n-s+1,n}\} \supset \{n, \ldots, n-k+1\}\} \\
& + \Pr\{R_{n-s+1,n} \leq n-k \text{ and } \{R_{n,n}, \ldots, R_{n-s+2,n}\} \supset \{n, \ldots, n-k+1\}\} .
\end{aligned}
\tag{2.5}
$$

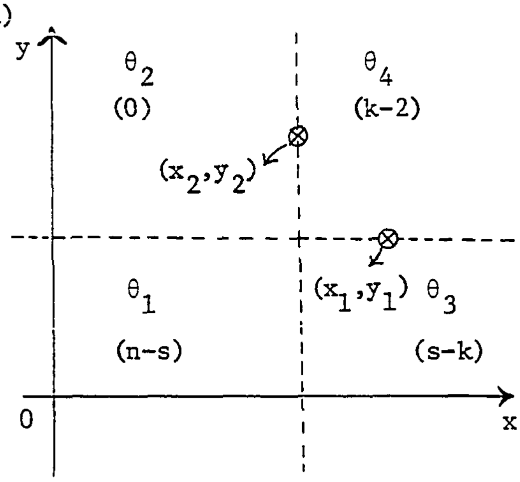
The manner in which the events on the right of (2.5) arise can best be seen in turn from the following graphs and diagrams:

(i)



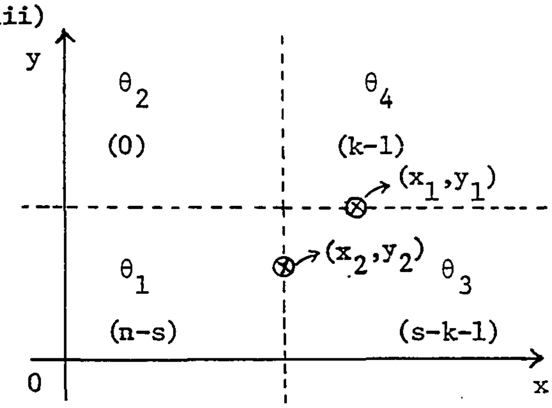
	$Y_i < Y_1$	$Y_i > Y_1$	
$X_i < X_1$	θ_1 n-s	θ_2 0	n-s
$X_i > X_1$	θ_3 s-k	θ_4 k-1	s-1
	n-k	k-1	n-1

(ii)



	$Y_i < Y_1$	$Y_i > Y_1$	
$X_i < X_2$	θ_1 n-s	θ_2 0	n-s
$X_i > X_2$	θ_3 s-k	θ_4 k-2	s-2
	n-k	k-2	n-2

(iii)



	$Y_i < Y_1$	$Y_i > Y_1$	
$X_i < X_2$	θ_1 n-s	θ_2 0	n-s
$X_i > X_2$	θ_3 s-k-1	θ_4 k-1	s-2
	n-k-1	k-1	n-2

The numbers in parentheses indicate the number of observations which fall into the θ_i regions, where the θ_i are defined in (1.3).

By conditioning on one of the (X_i,Y_i), say (X_1,Y_1), for case (i) and any two of the (X_i,Y_i), say (X_1,Y_1) and (X_2,Y_2), for cases (ii) and (iii), we have that, corresponding to (i), (ii), and (iii), respectively,

$$ {}_nP^{(1)}_{s:k} = \Pr\{R_{n-s+1,n} = n-k+1 \text{ and } \{R_{n,n}, \ldots, R_{n-s+2,n}\} \supset \{n, \ldots, n-k+2\}\} $$

$$ = \frac{n!}{(n-s)!(s-k)!(k-1)!1!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-s}(x,y)\theta_3^{s-k}(x,y)\theta_4^{k-1}(x,y)f(x,y)\,dxdy \tag{2.6} $$

for $1 \le k \le s \le n$,

$$ {}_nP^{(2)}_{s:k} = \Pr\{R_{n-s+1,n} \ge n-k+2 \text{ and } \{R_{n,n}, \ldots, R_{n-s+1,n}\} \supset \{n, \ldots, n-k+1\}\} $$

$$ = \frac{n!}{(n-s)!(s-k)!(k-2)!1!1!} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dx_2 \int_{y_1}^{\infty} dy_2 \int_{x_2}^{\infty} \theta_1^{n-s}(x_2,y_1) $$

$$\theta_3^{s-k}(x_2,y_1)\theta_4^{k-2}(x_2,y_1)f(x_1,y_1)f(x_2,y_2)dx_1 \tag{2.7}$$

for $2 \le k \le s \le n$, and

$$ {}_nP^{(3)}_{s:k} = \Pr\{R_{n-s+1,n} \le n-k \text{ and } \{R_{n,n}, \ldots, R_{n-s+2,n}\} \supset \{n, \ldots, n-k+1\}\}$$

$$= \frac{n!}{(n-s)!(s-k-1)!(k-1)!1!1!} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{y_1} dy_2 \int_{x_2}^{\infty} \theta_1^{n-s}(x_2,y_1)$$

$$\theta_3^{s-k-1}(x_2,y_1)\,\theta_4^{k-1}(x_2,y_1)f(x_1,y_1)f(x_2,y_2)dx_1$$

for $1 \le k < s \le n$. (2.8)

If we define g^+, g^-, h^+, and h^- as

$$\left.\begin{aligned} g^+(x,y) &\equiv \int_x^{\infty} f(u,y)du, \qquad g^-(x,y) \equiv \int_{-\infty}^{x} f(u,y)du\ , \\ h^+(x,y) &\equiv \int_y^{\infty} f(x,v)dv \quad \text{and } h^-(x,y) \equiv \int_{-\infty}^{y} f(x,v)dv\ , \end{aligned}\right\} \tag{2.9}$$

then we have simpler forms of ${}_nP^{(2)}_{s:k}$ and ${}_nP^{(3)}_{s:k}$, i.e.,

$${}_nP^{(2)}_{s:k} = \frac{n!}{(n-s)!(s-k)!(k-2)!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_1^{n-s}(x,y)\theta_3^{s-k}(x,y)$$

$$\theta_4^{k-2}(x,y)g^+(x,y)h^+(x,y)dxdy \tag{2.10}$$

and

$${}_nP^{(3)}_{s:k} = \frac{n!}{(n-s)!(s-k-1)!(k-1)!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_1^{n-s}(x,y)\theta_3^{s-k-1}(x,y)$$

$$\theta_4^{k-1}(x,y)g^+(x,y)h^-(x,y)dxdy. \tag{2.11}$$

Hence, we have

$$ {}_n\Pi_{s:k} = {}_nP^{(1)}_{s:k}1_{\{1\le k\le s\le n\}} + {}_nP^{(2)}_{s:k}1_{\{2\le k\le s\le n\}} $$

$$ + {}_nP^{(3)}_{s:k}1_{\{1\le k<s\le n\}} \qquad (2.12) $$

where $1_{\{\ \}}$ is the indicator function of the event $\{\ \}$.

The next lemma simplifies the expression (2.12) and is used for generating the probability ${}_n\Pi_{s:k}$.

Lemma 2.1: For $1 \le k \le s < n$, ${}_n\Pi_{s:k} = {}_nP^{(3)}_{s+1:k}$.

Proof: From the last term of equation (2.5) we have

$$ {}_nP^{(3)}_{s+1:k} = \Pr\{R_{n-s,n} \le n-k \text{ and } \{R_{n,n}, \ldots, R_{n-s+1,n}\} \supset \{n, \ldots, n-k+1\}\} . $$

But the right side of the above equation is equivalent to the probability $\Pr\{\{R_{n,n}, \ldots, R_{n-s+1,n}\} \supset \{n, \ldots, n-k+1\}\} = {}_n\Pi_{s:k}$ since the event $\{\{R_{n,n}, \ldots, R_{n-s+1,n}\} \supset \{n, \ldots, n-k+1\}\}$ implies the event $\{R_{n-s,n} \le n-k\}$. Hence, the lemma holds. Q.E.D.

Note that this lemma can be proved analytically with equations (2.6), (2.10), and (2.11) using Lemma 2.4 in the following subsection.

3. Properties of and relations among the probabilities ${}_n\Pi_{s:k}$ and ${}_nP^{(i)}_{s:k}$

We will study the relations among ${}_n\Pi_{s:k}$, ${}_nP^{(1)}_{s:k}$, ${}_nP^{(2)}_{s:k}$ and ${}_nP^{(3)}_{s:k}$ as well as their properties when the underlying p.d.f. f(x,y) is symmetric with respect to x and y and also symmetric about zero in each of x and y .

Lemma 2.2: ${}_n\Pi_{1:1} = {}_nP^{(1)}_{1:1} = \Pr\{R_{n,n} = n\}$.

Proof: From (2.12) we have

$$
\begin{aligned}
{}_n\Pi_{1:1} &= {}_nP^{(1)}_{1:1} \\
&= n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-1}(x,y) f(x,y)\,dxdy \ .
\end{aligned}
$$

But from equation (1.4) with $r = n$ and $s = n$, we have

$$
\Pr\{R_{n,n} = n\} = n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-1}(x,y) f(x,y)\,dxdy \ . \qquad \text{Q.E.D.}
$$

Lemma 2.3: ${}_nP^{(1)}_{n:n} = \Pr\{R_{1,n} = 1\}$.

Proof: From (1.4) and (2.6)

$$
\begin{aligned}
{}_nP^{(1)}_{n:n} &= n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_4^{n-1}(x,y) f(x,y)\,dxdy \\
&= \Pr\{R_{1,n} = 1\} \ . \qquad \text{Q.E.D.}
\end{aligned}
$$

The following lemma is given in order to prove the remaining lemmas in this subsection.

Lemma 2.4: For the absolutely continuous joint c.d.f. $F(x,y)$, with p.d.f. $f(x,y)$, we have the following equalities:

(i) $g^+(x,y) + g^-(x,y) = f_Y(y) = g^+(-\infty,y) = g^-(\infty,y)$

(ii) $h^+(x,y) + h^-(x,y) = f_X(x) = h^+(x,-\infty) = h^-(x,\infty)$

(iii) $\frac{\partial}{\partial x} g^-(x,y) = \frac{\partial}{\partial y} h^-(x,y) = -\frac{\partial}{\partial x} g^+(x,y) = -\frac{\partial}{\partial y} h^+(x,y) = f(x,y)$

(iv) $\frac{\partial}{\partial x}\theta_1(x,y) = h^-(x,y) = -\frac{\partial}{\partial x}\theta_3(x,y)$

$\frac{\partial}{\partial x}\theta_2(x,y) = h^+(x,y) = -\frac{\partial}{\partial x}\theta_4(x,y)$

$\frac{\partial}{\partial y}\theta_1(x,y) = g^-(x,y) = -\frac{\partial}{\partial y}\theta_2(x,y)$

$\frac{\partial}{\partial y}\theta_3(x,y) = g^+(x,y) = -\frac{\partial}{\partial y}\theta_4(x,y)$

(v) $F_Y(y) = \theta_1(\infty,y) = \theta_3(-\infty,y) = 1-\theta_2(\infty,y) = 1-\theta_4(-\infty,y)$

$F_X(x) = \theta_1(x,\infty) = \theta_2(x,-\infty) = 1-\theta_3(x,\infty) = 1-\theta_4(x,-\infty)$

where $f_Y(y)$ and $f_X(x)$ are the marginal p.d.f., and $F_Y(y)$ and $F_X(x)$ are the marginal c.d.f. of Y and X respectively.

Proof: (i), (ii), and (v) are the direct results from the definition (1.3) and (2.7). For (iii) and (iv) see Royden (1968). Q.E.D.

Lemma 2.5: For $1 \leq k < n$, ${}_nP^{(2)}_{n:k+1} = {}_nP^{(1)}_{n:k} + {}_nP^{(2)}_{n:k}$.

Proof: This lemma can be proved using (2.5) but we now prove this analytically.

$${}_nP^{(2)}_{n:k+1} = \frac{n!}{(n-k-1)!(k-1)!}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \theta_3^{n-k-1}(x,y)\theta_4^{k-1}(x,y)g^+(x,y)h^+(x,y)$$

$$= \frac{n!}{(n-k)!(k-1)!}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy[\frac{\partial}{\partial y}\theta_3^{n-k}(x,y)]\theta_4^{k-1}(x,y)h^+(x,y)$$

$$= \frac{n!}{(n-k)!\,(k-1)!} \int_{-\infty}^{\infty} dx \left\{ \theta_3^{n-k}(x,y)\theta_4^{k-1}(x,y)h^+(x,y) \Big|_{y=-\infty}^{y=\infty} \right.$$

$$+ (k-1) \int_{-\infty}^{\infty} dy\ \theta_3^{n-k}(x,y)\theta_4^{k-2}(x,y)g^+(x,y)h^+(x,y)$$

$$\left. + \int_{-\infty}^{\infty} dy\ \theta_3^{n-k}(x,y)\theta_4^{k-1}(x,y)f(x,y) \right\}$$

$$= \frac{n!}{(n-k)!\,(k-2)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_3^{n-k}(x,y)\theta_4^{k-2}(x,y)g^+(x,y)h^+(x,y)\,dxdy$$

$$+ \frac{n!}{(n-k)!\,(k-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_3^{n-k}(x,y)\theta_4^{k-1}(x,y)\,dxdy$$

$$= {}_nP_{n:k}^{(2)} + {}_nP_{n:k}^{(1)}$$

since $h^+(x,\infty) = 0$ and $\theta_3(x,-\infty) = 0$. Q.E.D.

The following lemma is used to reduce the number of computations of the probability ${}_n\Pi_{s:k}$ when the joint p.d.f. $f(x,y)$ satisfies the symmetry properties:

$$f(x,y) = f(y,x) = f(-x,-y) \ . \tag{2.13}$$

Note that from (2.13), we have

$$f_X(a) = f_Y(a) = f_X(-a) = f_Y(-a) \ . \tag{2.14}$$

Lemma 2.6: Suppose the p.d.f. $f(x,y)$ satisfies (2.13). Then

$${}_n\Pi_{s:k} = {}_n\Pi_{n-k:n-s} \quad \text{for all} \quad 1 \le k \le s \le n \ .$$

Proof: We have the following equality of the events:

$$\{\{R_{n,n}, R_{n-1,n}, \ldots, R_{n-s+1,n}\} \supset \{n, n-1, \ldots, n-k+1\}\}$$

$$= \{\{R_{1,n}, R_{2,n}, \ldots, R_{n-s,n}\} \subset \{1, 2, \ldots, n-k\}\} \ .$$

Because of symmetry of X and Y , we have that the event of the right side is

$$\{\{R_{n,n}, \ldots, R_{k+1,n}\} \supset \{n, \ldots, s+1\}\} .$$

Hence, we have ${}_n\Pi_{s:k} = {}_n\Pi_{n-k:n-s}$ by definition of ${}_n\Pi_{s:k}$. Q.E.D.

Note that we can also prove above lemma using Lemma 2.1, and the following properties under the symmetry assumptions (2.13):

$$\text{(i)} \quad \theta_1(-x,-y) = \int_{-\infty}^{-y} dv \int_{-\infty}^{-x} du\ f(u,v)$$

$$= \int_{-\infty}^{-y} dv \int_{-\infty}^{-x} du\ f(-u,-v)$$

$$= \int_{y}^{\infty} dz \int_{x}^{\infty} dw\ f(w,z)\,(w = -u \text{ and } z = -v)$$

$$= \theta_4(x,y) \tag{2.15}$$

$$\text{(ii)} \quad \theta_3(-x,-y) = \theta_2(x,y) = \theta_3(y,x) \tag{2.16}$$

$$\text{(iii)} \quad \theta_1(x,y) = \theta_1(y,x) \quad \text{and} \quad \theta_4(x,y) = \theta_4(y,x) \tag{2.17}$$

$$\text{(iv)} \quad g^+(-x,-y) = g^-(x,y) = h^-(y,x) \tag{2.18}$$

$$\text{(v)} \quad h^-(-x,-y) = h^+(x,y) = g^+(y,x) . \tag{2.19}$$

Lemma 2.7: If the p.d.f. $f(x,y)$ satisfies (2.13), then

$${}_nP^{(1)}_{s:k} = {}_nP^{(1)}_{n-k+1:n-s+1} \quad \text{for all} \quad 1 \leq k \leq s \leq n .$$

Proof: From the relations (1.5), (1.6), and a graph similar to (i) of previous subsection, we have

$${}_nP^{(1)}_{s:k} = \Pr\{R_{n-s+1,n} = n-k+1 \text{ and } \{R_{n,n}, \ldots, R_{n-s+2,n}\} \supset \{n, \ldots, n-k+2\}\}$$

$$= \Pr\{R_{s,n} = k \text{ and } \{R_{1,n}, \ldots, R_{s-1,n}\} \supset \{1, 2, \ldots, k-1\}\}$$

$$= \frac{n!}{(k-1)!(s-k)!(n-s)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{k-1}(x,y)\theta_2^{s-k}(x,y)\theta_4^{n-s}(x,y)f(x,y)dxdy.$$

Using (2.15)-(2.17), we then have

$${}_nP_{s:k}^{(1)} = \frac{n!}{(k-1)!(s-k)!(n-s)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{k-1}(y,x)\theta_3^{s-k}(y,x)\theta_4^{n-s}(y,x)f(y,x)dxdy$$

$$= {}_nP_{n-k+1:n-s+1}^{(1)} \;. \qquad \text{Q.E.D.}$$

Computation of ${}_n\Pi_{s:k}$

For any absolutely continuous c.d.f. $F(x,y)$ with p.d.f. $f(x,y)$, the probability ${}_n\Pi_{s:k}$ can be computed using Lemma 2.1, i.e.,

$${}_n\Pi_{s:k} = \frac{n!}{(n-s-1)!(s-k)!(k-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-s-1}(x,y)\theta_3^{s-k}(x,y)\theta_4^{k-1}(x,y) g^{+}(x,y)h^{-}(x,y)dxdy \,, \tag{2.20}$$

for $1 \le k \le s < n$. When $s = n$ we know that ${}_n\Pi_{n:k} = 1$ for all $1 \le k \le n$. Usually these multiple integrals can be evaluated by Gaussian quadrature. Since θ_i, g^{+}, and h^{-} are themselves integrals, we need to transform the variables to get a simple form of integrand or to factor out the weight function of appropriate quadrature formulae.

When the joint p.d.f. has the symmetry properties (2.13), we can reduce considerably the number of computations in the numerical multiple integration.

E. Numerical Results for the Bivariate Normal Case

1. Numerical evaluation of the probabilities

The rank $R_{r,n}$ of a concomitant of an order statistic is location and scale invariant with respect to both X and Y . Hence, we may take $F(x,y)$ and $f(x,y)$ to refer to the standardized variates. In this section, we will evaluate the probability ${}_n\Pi_{s:k}$ when (X,Y) has the standard bivariate normal p.d.f.

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}, |\rho| < 1 ,$$

which has the symmetry properties (2.13).

The probability of selecting the largest Y_i using the theory of the rank of a concomitant is found in David et al. (1977). Since $\Pr\{R_{1,n} = 1\} = \Pr\{R_{n,n} = n\} = {}_n\Pi_{1:1}$, the probability ${}_n\Pi_{1:1}$ is a special case of their table of $\Pr\{R_{r,n} = s\}$. Since the case when $\rho \cong 1$ is of special interest in applications, Table 2.1 has been constructed to give ${}_n\Pi_{1:1}$ for $\rho = 0.1(0.1)0.9$, 0.95, 0.99, 0.995, 0.999 and $n = 2, 3, \ldots, 100$. Also, Graph 2.1 shows ${}_n\Pi_{1:1}(\rho)$ for $-1 \leq \rho \leq 1$ and Graph 2.2 is a magnified version of Graph 2.1 for $0.8 \leq \rho \leq 1$. We see that the probability ${}_n\Pi_{1:1}$ is not necessarily high even though ρ is very close to 1, the slope of the graph being very steep near $\rho = 1$. For selected values of $P = {}_n\Pi_{1:1}(\rho)$, Table 2.2 shows the corresponding ρ values.

Table 2.1 ${}_n\pi_{1:1} = \Pr\left\{ \text{rank } Y_{(n:n)} = n \right\}$ for $(X,Y) \sim BVN(0,0,1,1,\rho)$

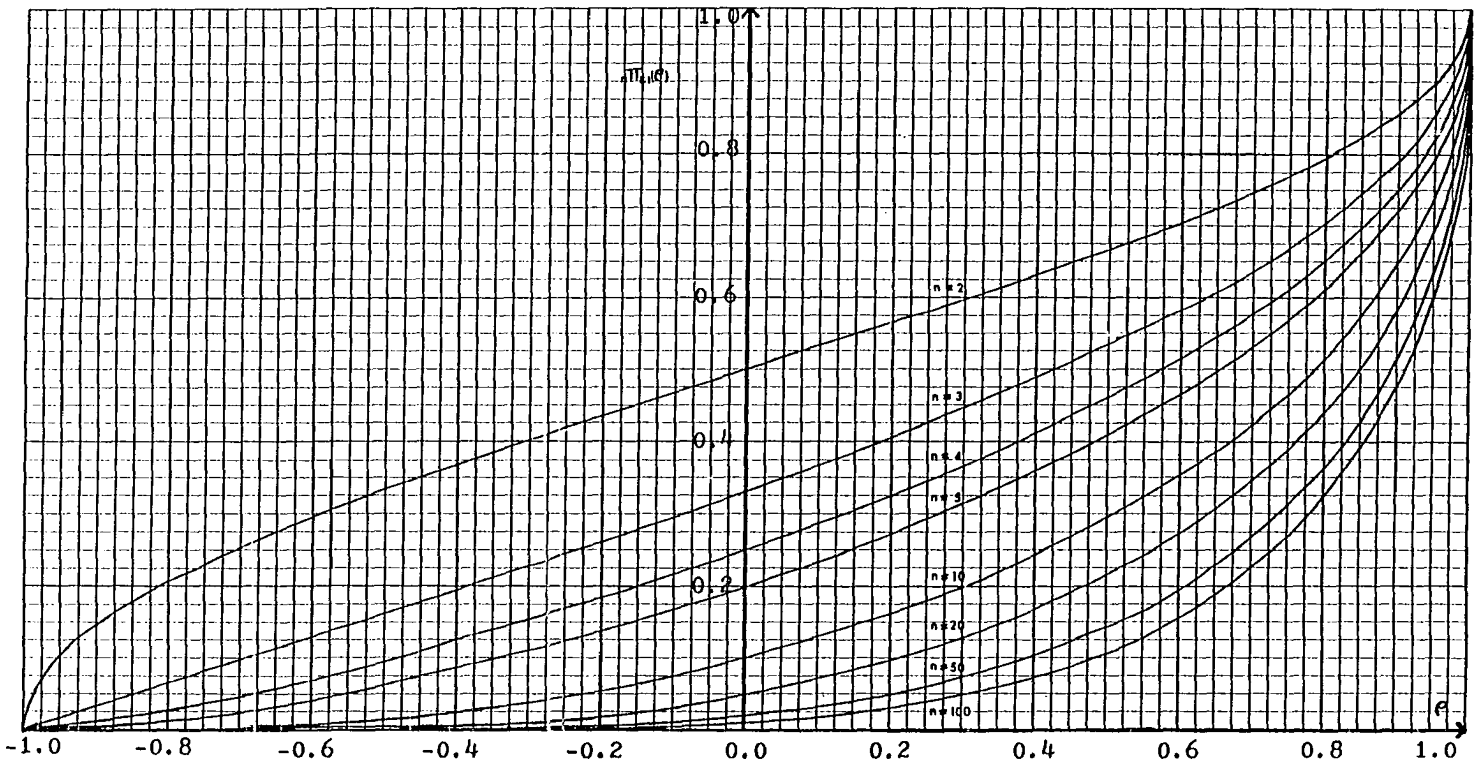
$n \backslash \rho$	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	0.990	0.995	0.999
2	0.5319	0.5641	0.5970	0.6310	0.6667	0.7048	0.7468	0.7952	0.8564	0.8989	0.9550	0.9681	0.9855
3	0.3698	0.4076	0.4474	0.4894	0.5346	0.5841	0.6398	0.7054	0.7908	0.8514	0.9330	0.9525	0.9784
4	0.2864	0.3251	0.3665	0.4112	0.4601	0.5145	0.5768	0.6514	0.7503	0.8216	0.9190	0.9424	0.9737
5	0.2352	0.2733	0.3149	0.3604	0.4109	0.4678	0.5338	0.6140	0.7218	0.8004	0.9089	0.9352	0.9704
6	0.2004	0.2375	0.2786	0.3242	0.3753	0.4337	0.5021	0.5860	0.7001	0.7841	0.9010	0.9295	0.9677
7	0.1751	0.2112	0.2515	0.2968	0.3482	0.4073	0.4773	0.5639	0.6828	0.7711	0.8947	0.9249	0.9656
8	0.1559	0.1908	0.2303	0.2752	0.3265	0.3861	0.4571	0.5458	0.6685	0.7602	0.8894	0.9211	0.9638
9	0.1407	0.1746	0.2133	0.2576	0.3087	0.3686	0.4404	0.5306	0.6564	0.7510	0.8848	0.9178	0.9623
10	0.1284	0.1613	0.1992	0.2430	0.2938	0.3537	0.4261	0.5176	0.6459	0.7429	0.8809	0.9149	0.9610
11	0.1182	0.1502	0.1873	0.2305	0.2810	0.3409	0.4137	0.5062	0.6368	0.7359	0.8773	0.9124	0.9598
12	0.1097	0.1407	0.1771	0.2198	0.2700	0.3298	0.4028	0.4962	0.6286	0.7296	0.8742	0.9101	0.9587
13	0.1023	0.1326	0.1683	0.2104	0.2602	0.3199	0.3932	0.4873	0.6213	0.7239	0.8714	0.9080	0.9577
14	0.0960	0.1255	0.1605	0.2021	0.2516	0.3111	0.3845	0.4792	0.6147	0.7188	0.8688	0.9061	0.9569
15	0.0905	0.1192	0.1537	0.1947	0.2438	0.3032	0.3767	0.4719	0.6086	0.7141	0.8664	0.9044	0.9561
16	0.0856	0.1137	0.1475	0.1881	0.2368	0.2960	0.3695	0.4652	0.6031	0.7097	0.8642	0.9028	0.9553
17	0.0813	0.1087	0.1420	0.1821	0.2304	0.2894	0.3630	0.4590	0.5980	0.7057	0.8622	0.9014	0.9546
18	0.0774	0.1043	0.1370	0.1766	0.2246	0.2834	0.3570	0.4533	0.5932	0.7020	0.8603	0.9000	0.9540
19	0.0739	0.1002	0.1324	0.1716	0.2193	0.2779	0.3514	0.4481	0.5888	0.6985	0.8586	0.8987	0.9534
20	0.0707	0.0965	0.1282	0.1670	0.2144	0.2727	0.3463	0.4431	0.5847	0.6953	0.8569	0.8975	0.9529
21	0.0678	0.0931	0.1244	0.1628	0.2098	0.2679	0.3414	0.4385	0.5808	0.6922	0.8554	0.8964	0.9524
22	0.0652	0.0900	0.1208	0.1588	0.2055	0.2635	0.3369	0.4342	0.5771	0.6893	0.8539	0.8953	0.9519
23	0.0627	0.0871	0.1175	0.1551	0.2015	0.2593	0.3327	0.4301	0.5737	0.6866	0.8525	0.8943	0.9514
24	0.0605	0.0845	0.1145	0.1517	0.1978	0.2554	0.3287	0.4263	0.5704	0.6840	0.8511	0.8933	0.9509
25	0.0584	0.0820	0.1116	0.1485	0.1943	0.2517	0.3249	0.4226	0.5673	0.6815	0.8498	0.8923	0.9504
26	0.0565	0.0797	0.1089	0.1455	0.1910	0.2482	0.3213	0.4191	0.5643	0.6791	0.8486	0.8914	0.9500
27	0.0547	0.0775	0.1064	0.1427	0.1879	0.2449	0.3179	0.4158	0.5615	0.6769	0.8474	0.8905	0.9495
28	0.0530	0.0755	0.1041	0.1400	0.1850	0.2417	0.3147	0.4127	0.5588	0.6747	0.8462	0.8897	0.9491
29	0.0515	0.0736	0.1018	0.1374	0.1822	0.2387	0.3116	0.4097	0.5562	0.6727	0.8451	0.8888	0.9487
30	0.0500	0.0718	0.0997	0.1350	0.1795	0.2359	0.3087	0.4069	0.5538	0.6707	0.8441	0.8880	0.9483
31	0.0486	0.0701	0.0977	0.1328	0.1770	0.2332	0.3059	0.4041	0.5514	0.6688	0.8431	0.8873	0.9479
32	0.0473	0.0685	0.0959	0.1306	0.1746	0.2306	0.3032	0.4015	0.5491	0.6670	0.8421	0.8866	0.9475
33	0.0461	0.0670	0.0941	0.1286	0.1723	0.2281	0.3006	0.3990	0.5470	0.6652	0.8412	0.8859	0.9471
34	0.0450	0.0656	0.0924	0.1266	0.1701	0.2258	0.2982	0.3966	0.5449	0.6635	0.8403	0.8852	0.9468
35	0.0439	0.0642	0.0907	0.1247	0.1680	0.2235	0.2958	0.3942	0.5429	0.6619	0.8394	0.8846	0.9464
36	0.0428	0.0629	0.0892	0.1229	0.1660	0.2213	0.2935	0.3920	0.5409	0.6604	0.8386	0.8840	0.9461
37	0.0418	0.0617	0.0877	0.1212	0.1641	0.2192	0.2914	0.3899	0.5390	0.6589	0.8379	0.8834	0.9459
38	0.0409	0.0605	0.0863	0.1196	0.1623	0.2172	0.2893	0.3878	0.5372	0.6574	0.8371	0.8829	0.9456
39	0.0400	0.0594	0.0849	0.1180	0.1605	0.2153	0.2872	0.3858	0.5355	0.6560	0.8364	0.8824	0.9454
40	0.0392	0.0584	0.0836	0.1165	0.1588	0.2134	0.2853	0.3838	0.5338	0.6547	0.8358	0.8819	0.9452

Table 2.1 (continued)

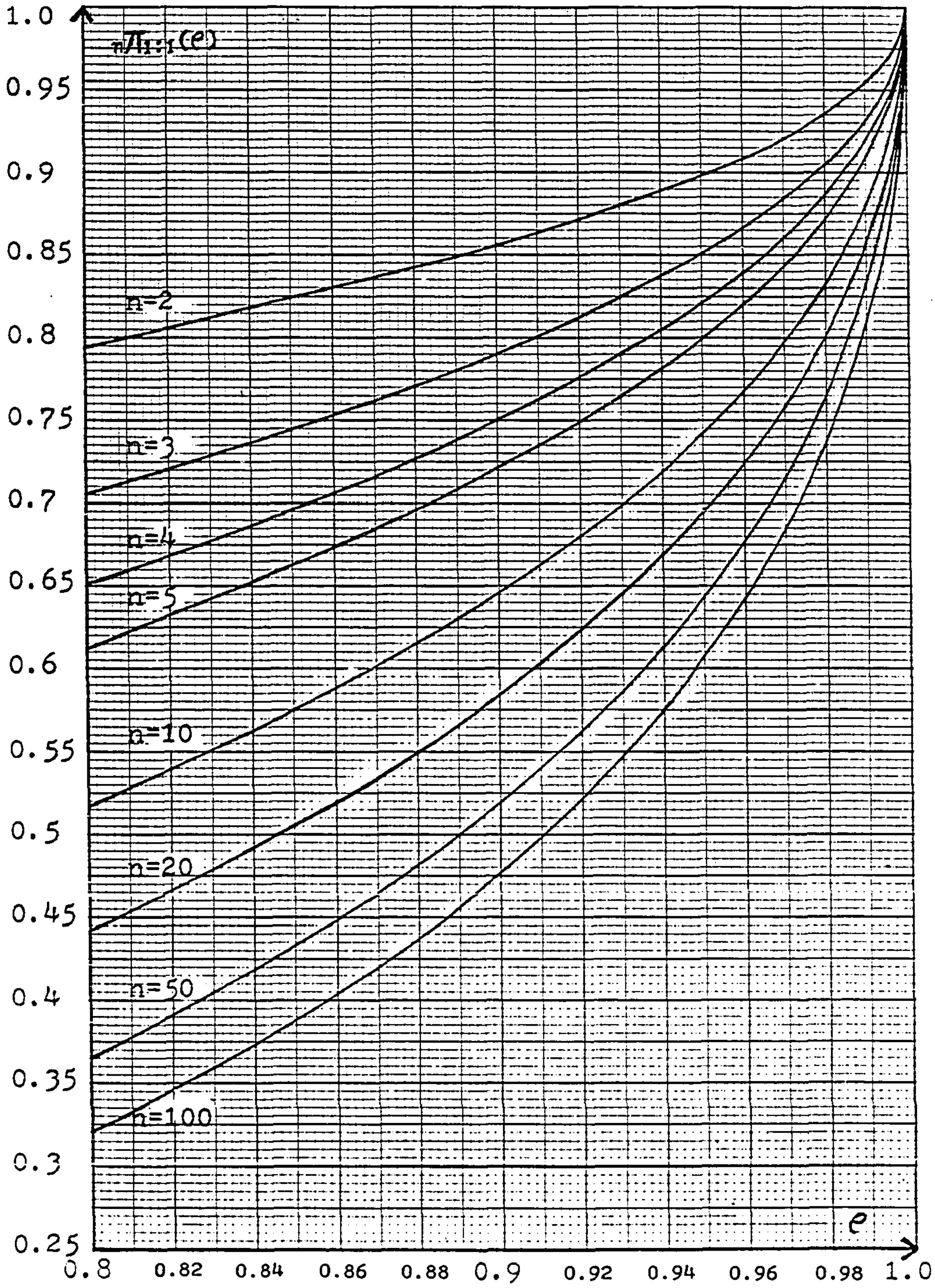
n\ρ	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	0.990	0.995	0.999
41	0.0383	0.0573	0.0824	0.1150	0.1571	0.2116	0.2834	0.3820	0.5322	0.6534	0.8351	0.8814	0.9450
42	0.0376	0.0563	0.0812	0.1136	0.1555	0.2099	0.2816	0.3801	0.5306	0.6521	0.8345	0.8810	0.9448
43	0.0368	0.0554	0.0801	0.1123	0.1540	0.2082	0.2798	0.3784	0.5290	0.6509	0.8339	0.8806	0.9446
44	0.0361	0.0545	0.0790	0.1110	0.1525	0.2066	0.2781	0.3767	0.5276	0.6497	0.8333	0.8802	0.9445
45	0.0354	0.0536	0.0779	0.1097	0.1511	0.2050	0.2764	0.3750	0.5261	0.6485	0.8328	0.8798	0.9444
46	0.0348	0.0528	0.0769	0.1085	0.1497	0.2035	0.2748	0.3734	0.5247	0.6474	0.8322	0.8795	0.9443
47	0.0342	0.0520	0.0759	0.1073	0.1484	0.2020	0.2732	0.3719	0.5233	0.6463	0.8317	0.8791	0.9442
48	0.0335	0.0512	0.0749	0.1062	0.1471	0.2006	0.2717	0.3703	0.5220	0.6452	0.8312	0.8788	0.9441
49	0.0330	0.0505	0.0740	0.1051	0.1458	0.1992	0.2702	0.3689	0.5207	0.6442	0.8307	0.8785	0.9440
50	0.0324	0.0498	0.0731	0.1041	0.1446	0.1978	0.2688	0.3674	0.5194	0.6432	0.8302	0.8782	0.9439
51	0.0319	0.0491	0.0723	0.1030	0.1434	0.1965	0.2674	0.3660	0.5182	0.6422	0.8298	0.8779	0.9439
52	0.0314	0.0484	0.0714	0.1020	0.1423	0.1952	0.2660	0.3646	0.5170	0.6412	0.8293	0.8776	0.9438
53	0.0309	0.0477	0.0706	0.1011	0.1412	0.1940	0.2647	0.3633	0.5158	0.6403	0.8289	0.8773	0.9437
54	0.0304	0.0471	0.0698	0.1001	0.1401	0.1928	0.2634	0.3620	0.5146	0.6393	0.8284	0.8770	0.9437
55	0.0299	0.0465	0.0691	0.0992	0.1390	0.1916	0.2621	0.3607	0.5135	0.6384	0.8279	0.8767	0.9436
56	0.0295	0.0459	0.0683	0.0983	0.1380	0.1904	0.2609	0.3595	0.5124	0.6375	0.8275	0.8764	0.9435
57	0.0290	0.0453	0.0676	0.0974	0.1370	0.1893	0.2597	0.3582	0.5113	0.6366	0.8271	0.8761	0.9435
58	0.0286	0.0448	0.0669	0.0966	0.1360	0.1882	0.2585	0.3570	0.5102	0.6357	0.8266	0.8758	0.9434
59	0.0282	0.0442	0.0662	0.0958	0.1350	0.1871	0.2573	0.3558	0.5091	0.6348	0.8262	0.8755	0.9433
60	0.0278	0.0437	0.0656	0.0950	0.1341	0.1861	0.2562	0.3547	0.5081	0.6340	0.8257	0.8751	0.9432
61	0.0274	0.0432	0.0649	0.0942	0.1332	0.1850	0.2551	0.3536	0.5071	0.6331	0.8253	0.8748	0.9431
62	0.0270	0.0427	0.0643	0.0934	0.1323	0.1840	0.2540	0.3524	0.5061	0.6323	0.8248	0.8745	0.9430
63	0.0267	0.0422	0.0637	0.0927	0.1314	0.1831	0.2529	0.3514	0.5051	0.6315	0.8244	0.8742	0.9429
64	0.0263	0.0417	0.0631	0.0919	0.1305	0.1821	0.2519	0.3503	0.5041	0.6307	0.8239	0.8739	0.9427
65	0.0260	0.0413	0.0625	0.0912	0.1297	0.1811	0.2509	0.3492	0.5031	0.6299	0.8235	0.8736	0.9426
66	0.0257	0.0408	0.0619	0.0905	0.1289	0.1802	0.2499	0.3482	0.5022	0.6291	0.8230	0.8732	0.9425
67	0.0253	0.0404	0.0614	0.0899	0.1281	0.1793	0.2489	0.3472	0.5013	0.6283	0.8226	0.8729	0.9423
68	0.0250	0.0400	0.0608	0.0892	0.1273	0.1784	0.2479	0.3462	0.5004	0.6276	0.8222	0.8726	0.9421
69	0.0247	0.0396	0.0603	0.0885	0.1265	0.1775	0.2469	0.3452	0.4995	0.6268	0.8217	0.8722	0.9420
70	0.0244	0.0392	0.0598	0.0879	0.1258	0.1767	0.2460	0.3442	0.4986	0.6261	0.8213	0.8719	0.9418
71	0.0241	0.0388	0.0593	0.0873	0.1250	0.1758	0.2451	0.3433	0.4977	0.6253	0.8208	0.8715	0.9416
72	0.0239	0.0384	0.0588	0.0867	0.1243	0.1750	0.2442	0.3424	0.4968	0.6246	0.8204	0.8712	0.9414
73	0.0236	0.0380	0.0583	0.0861	0.1236	0.1742	0.2433	0.3414	0.4960	0.6239	0.8199	0.8708	0.9412
74	0.0233	0.0376	0.0578	0.0855	0.1229	0.1734	0.2424	0.3405	0.4952	0.6232	0.8195	0.8704	0.9410
75	0.0230	0.0373	0.0574	0.0849	0.1222	0.1726	0.2416	0.3396	0.4943	0.6225	0.8190	0.8701	0.9408
76	0.0228	0.0369	0.0569	0.0843	0.1216	0.1719	0.2407	0.3388	0.4935	0.6218	0.8186	0.8697	0.9405
77	0.0225	0.0366	0.0565	0.0838	0.1209	0.1711	0.2399	0.3379	0.4927	0.6211	0.8181	0.8693	0.9403
78	0.0223	0.0362	0.0560	0.0832	0.1203	0.1704	0.2391	0.3371	0.4919	0.6204	0.8177	0.8690	0.9401
79	0.0221	0.0359	0.0556	0.0827	0.1196	0.1696	0.2383	0.3362	0.4911	0.6197	0.8172	0.8686	0.9398
80	0.0218	0.0356	0.0552	0.0822	0.1190	0.1689	0.2375	0.3354	0.4904	0.6191	0.8168	0.8682	0.9396

Table 2.1 (continued)

n\ρ	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	0.990	0.995	0.999
81	0.0216	0.0352	0.0548	0.0817	0.1184	0.1682	0.2367	0.3346	0.4896	0.6184	0.8164	0.8679	0.9393
82	0.0214	0.0349	0.0544	0.0812	0.1178	0.1675	0.2359	0.3338	0.4889	0.6178	0.8160	0.8675	0.9391
83	0.0212	0.0346	0.0540	0.0807	0.1172	0.1668	0.2352	0.3330	0.4882	0.6172	0.8155	0.8671	0.9388
84	0.0209	0.0343	0.0536	0.0802	0.1166	0.1662	0.2345	0.3322	0.4874	0.6165	0.8151	0.8668	0.9385
85	0.0207	0.0340	0.0532	0.0797	0.1160	0.1655	0.2337	0.3315	0.4867	0.6159	0.8147	0.8664	0.9383
86	0.0205	0.0338	0.0528	0.0792	0.1155	0.1649	0.2330	0.3307	0.4860	0.6153	0.8143	0.8661	0.9380
87	0.0203	0.0335	0.0525	0.0788	0.1149	0.1642	0.2323	0.3300	0.4853	0.6147	0.8139	0.8657	0.9377
88	0.0201	0.0332	0.0521	0.0783	0.1144	0.1636	0.2316	0.3293	0.4846	0.6142	0.8135	0.8654	0.9375
89	0.0199	0.0329	0.0517	0.0779	0.1138	0.1630	0.2309	0.3285	0.4840	0.6136	0.8131	0.8650	0.9372
90	0.0198	0.0327	0.0514	0.0775	0.1133	0.1624	0.2302	0.3278	0.4833	0.6130	0.8127	0.8647	0.9369
91	0.0196	0.0324	0.0510	0.0770	0.1128	0.1618	0.2296	0.3271	0.4827	0.6125	0.8123	0.8643	0.9367
92	0.0194	0.0321	0.0507	0.0766	0.1123	0.1612	0.2289	0.3265	0.4820	0.6119	0.8119	0.8640	0.9364
93	0.0192	0.0319	0.0504	0.0762	0.1118	0.1606	0.2283	0.3258	0.4814	0.6114	0.8116	0.8637	0.9361
94	0.0190	0.0317	0.0501	0.0758	0.1113	0.1600	0.2276	0.3251	0.4808	0.6108	0.8112	0.8634	0.9359
95	0.0189	0.0314	0.0497	0.0754	0.1108	0.1595	0.2270	0.3245	0.4802	0.6103	0.8109	0.8630	0.9356
96	0.0187	0.0312	0.0494	0.0750	0.1103	0.1589	0.2264	0.3238	0.4795	0.6098	0.8105	0.8627	0.9354
97	0.0185	0.0309	0.0491	0.0746	0.1099	0.1584	0.2258	0.3232	0.4790	0.6093	0.8102	0.8624	0.9351
98	0.0184	0.0307	0.0488	0.0742	0.1094	0.1578	0.2252	0.3225	0.4784	0.6088	0.8098	0.8621	0.9349
99	0.0182	0.0305	0.0485	0.0738	0.1089	0.1573	0.2246	0.3219	0.4778	0.6083	0.8095	0.8618	0.9346
100	0.0181	0.0303	0.0482	0.0735	0.1085	0.1568	0.2240	0.3213	0.4772	0.6078	0.8092	0.8616	0.9344



Graph 2.1 $_n\Pi_{1,1}(\rho)$ for $-1 \leq \rho \leq 1$



Graph 2.2 Magnified graph of $_n\Pi_{1:1}(\rho)$ for $0.8<\rho<1.0$

Table 2.2 ρ-value satisfying $_n\Pi_{1:1}(\rho) = P$

n\P	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95
2	-	-	-	-	0.0	.3089	.5878	.8089	.9510	.9877
3	-	-	-	.1802	.4241	.6299	.7925	.9087	.9775	.9945
4	-	-	.1359	.3757	.5745	.7334	.8536	.9367	.9846	.9962
5	-	0.0	.2652	.4794	.6507	.7841	.8828	.9497	.9879	.9970
6	-	.0988	.3483	.5439	.6972	.8143	.8999	.9573	.9898	.9975
7	-	.1702	.4066	.5884	.7287	.8346	.9112	.9623	.9910	.9978
8	-	.2243	.4500	.6210	.7515	.8491	.9193	.9659	.9919	.9980
9	-	.2671	.4839	.6461	.7690	.8602	.9255	.9685	.9925	.9981
10	0.0	.3019	.5111	.6662	.7827	.8690	.9303	.9706	.9930	.9983
11	.0349	.3309	.5335	.6827	.7941	.8761	.9342	.9723	.9934	.9984
12	.0650	.3556	.5524	.6965	.8035	.8820	.9374	.9737	.9938	.9984
13	.0914	.3769	.5687	.7083	.8115	.8870	.9402	.9749	.9941	.9985
14	.1145	.3955	.5829	.7185	.8184	.8913	.9426	.9760	.9943	.9986
15	.1351	.4118	.5953	.7274	.8245	.8951	.9446	.9768	.9946	.9986
16	.1536	.4264	.6062	.7353	.8298	.8984	.9464	.9776	.9948	.9987
17	.1704	.4395	.6160	.7423	.8345	.9013	.9480	.9783	.9950	.9987
18	.1856	.4514	.6249	.7486	.8387	.9039	.9494	.9789	.9951	.9988
19	.1994	.4621	.6329	.7542	.8425	.9062	.9506	.9794	.9952	.9988
20	.2121	.4720	.6401	.7594	.8459	.9083	.9517	.9799	.9953	.9989
21	.2237	.4810	.6467	.7640	.8490	.9101	.9527	.9803	.9954	.9989
22	.2345	.4893	.6528	.7683	.8518	.9118	.9536	.9806	.9955	.9989
23	.2446	.4969	.6584	.7722	.8544	.9134	.9544	.9809	.9955	.9989
24	.2540	.5040	.6636	.7758	.8568	.9148	.9552	.9812	.9956	.9989
25	.2628	.5106	.6684	.7792	.8589	.9161	.9558	.9815	.9956	.9989
26	.2711	.5167	.6729	.7823	.8610	.9174	.9565	.9817	.9956	.9989
27	.2788	.5225	.6772	.7853	.8630	.9185	.9571	.9819	.9957	.9989
28	.2862	.5282	.6812	.7881	.8648	.9196	.9576	.9821	.9957	.9989
29	.2931	.5334	.6850	.7907	.8665	.9207	.9582	.9823	.9957	.9989
30	.2997	.5383	.6886	.7932	.8682	.9217	.9587	.9825	.9957	.9989
31	.3059	.5430	.6920	.7956	.8697	.9226	.9592	.9827	.9958	.9989
32	.3118	.5475	.6952	.7979	.8712	.9235	.9597	.9829	.9958	.9989
33	.3175	.5517	.6983	.8000	.8727	.9244	.9602	.9831	.9958	.9989
34	.3230	.5556	.7013	.8021	.8741	.9253	.9606	.9833	.9959	.9989
35	.3282	.5596	.7041	.8041	.8754	.9261	.9611	.9835	.9959	.9989
36	.3333	.5634	.7069	.8060	.8767	.9269	.9615	.9837	.9959	.9989
37	.3382	.5670	.7095	.8078	.8780	.9277	.9620	.9839	.9960	.9989
38	.3429	.5705	.7121	.8096	.8792	.9285	.9624	.9841	.9960	.9989
39	.3474	.5739	.7145	.8114	.8803	.9292	.9628	.9843	.9961	.9990
40	.3518	.5772	.7169	.8131	.8815	.9299	.9632	.9845	.9961	.9990
41	.3561	.5803	.7192	.8147	.8826	.9306	.9636	.9847	.9962	.9990
42	.3602	.5834	.7214	.8163	.8836	.9313	.9640	.9849	.9963	.9990
43	.3642	.5864	.7235	.8178	.8847	.9320	.9644	.9851	.9964	.9991
44	.3681	.5892	.7256	.8193	.8857	.9326	.9648	.9853	.9964	.9991
45	.3718	.5920	.7276	.8207	.8866	.9332	.9652	.9855	.9965	.9991
46	.3755	.5947	.7296	.8221	.8876	.9339	.9656	.9857	.9966	.9992
47	.3790	.5973	.7315	.8234	.8885	.9344	.9659	.9859	.9966	.9992
48	.3824	.5997	.7333	.8247	.8894	.9350	.9663	.9861	.9967	.9992
49	.3857	.6022	.7351	.8259	.8902	.9356	.9666	.9863	.9968	.9993
50	.3890	.6045	.7368	.8271	.8910	.9361	.9669	.9865	.9968	.9993

In general, the numerical evaluation of ${}_n\Pi_{s:k}$ is not easy if we want considerable accuracy (e.g., to four decimal places). This can be done by using triple Gaussian quadrature with 64 abscissas. Table 2.3 shows the probability ${}_n\Pi_{s:k}(\rho)$ for various ρ, n, s, and k . Lemma 2.6 in Section D is used to reduce the number of computations.

When $0 \leq \rho < 1$, equation (2.20) can be simplified since

$$g^+(x,y) = \int_x^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \exp[-\frac{u^2-2\rho uy+y^2}{2(1-\rho^2)}]\, du$$

$$= \int_{\frac{x-\rho y}{\sqrt{1-\rho^2}}}^\infty \frac{1}{2\pi} \exp[-\frac{U^2}{2}] \exp[-\frac{y^2}{2}]\, dU$$

$$\text{(using } u = \rho y+\sqrt{1-\rho^2}\, U)$$

$$= \phi(y)\ [1 - \Phi(\frac{x-\rho y}{\sqrt{1-\rho^2}})] = \phi(y)\ \Phi(\frac{\rho y-x}{\sqrt{1-\rho^2}}) .$$

Similarly,

$$h^-(x,y) = \phi(x)\ \Phi(\frac{y-\rho x}{\sqrt{1-\rho^2}}) ,$$

and θ_i can be expressed as a single integral, for example,

$$\theta_4(x,y) = \int_x^\infty \phi(u)\ [1 - \Phi(\frac{y-\rho u}{\sqrt{1-\rho^2}})]\, du$$

$$= \int_x^\infty \phi(u)\ \Phi(\frac{\rho u-y}{\sqrt{1-\rho^2}})\, du$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and $\Phi(z) = \int_{-\infty}^z \phi(u)du$ (see Lemma 3.1 and Lemma 3.2).

Table 2.3 Probability ${}_n\pi_{s:k}(\rho)$ that the s objects out of n with the largest x-values contain the k objects with the largest y-values for $(X,Y) \sim BVN(0,0,1,1,\rho)$

n	s	k\\ρ 0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
3	1	1 .3698	.4076	.4474	.4894	.5346	.5841	.6398	.7054	.7908	.8514	.9330	.9525
	2	1 .7019	.7363	.7700	.8032	.8360	.8686	.9012	.9338	.9667	.9833	.9966	.9983
4	1	1 .2864	.3251	.3665	.4112	.4601	.5145	.5768	.6514	.7503	.8216	.9190	.9425
	2	1 .5455	.5913	.6374	.6842	.7318	.7806	.8310	.8836	.9392	.9687	.9935	.9967
		2 .1976	.2319	.2703	.3135	.3626	.4195	.4872	.5716	.6880	.7749	.8965	.9264
	3	1 .7843	.8168	.8476	.8767	.9040	.9295	.9528	.9734	.9902	.9965	.9997	.9999
5	1	1 .2352	.2733	.3149	.3604	.4109	.4678	.5338	.6140	.7218	.8004	.9089	.9352
	2	1 .4488	.4990	.5507	.6041	.6598	.7180	.7795	.8453	.9172	.9565	.9907	.9953
		2 .1248	.1537	.1874	.2271	.2741	.3307	.4009	.4920	.6236	.7252	.8719	.9086
	3	1 .6476	.6942	.7396	.7841	.8273	.8692	.9092	.9465	.9791	.9921	.9992	.9997
		2 .3474	.3979	.4520	.5099	.5722	.6396	.7132	.7947	.8873	.9398	.9868	.9933
	4	1 .8325	.8625	.8901	.9152	.9379	.9577	.9746	.9878	.9967	.9991	1.0000	1.0000
6	1	1 .2004	.2375	.2786	.3242	.3753	.4337	.5021	.5860	.7001	.7841	.9010	.9296
	2	1 .3828	.4346	.4889	.5461	.6064	.6707	.7397	.8149	.8991	.9463	.9883	.9941
		2 .0868	.1112	.1407	.1765	.2204	.2750	.3447	.4383	.5782	.6894	.8537	.8953
	3	1 .5535	.6070	.6650	.7139	.7672	.8201	.8722	.9225	.9684	.9877	.9988	.9996
		2 .2427	.2903	.3435	.4029	.4694	.5442	.6293	.7277	.8453	.9152	.9808	.9902
		3 .0666	.0874	.1131	.1453	.1858	.2375	.3055	.3993	.5438	.6616	.8392	.8847
	4	1 .7140	.7591	.8021	.8427	.8807	.9156	.9467	.9729	.9920	.9978	.9999	1.0000
		2 .4557	.5133	.5729	.6343	.6974	.7620	.8275	.8928	.9549	.9820	.9981	.9993
	5	1 .8639	.8915	.9163	.9381	.9570	.9727	.9851	.9939	.9987	.9998	1.0000	1.0000
7	1	1 .1751	.2112	.2515	.2968	.3482	.4073	.4773	.5639	.6828	.7711	.8947	.9250
	2	1 .3348	.3869	.4424	.5016	.5650	.6333	.7077	.7901	.8840	.9376	.9863	.9930
		2 .0643	.0851	.1111	.1436	.1845	.2366	.3050	.3992	.5439	.6618	.8393	.8848
	3	1 .4846	.5417	.5999	.6590	.7191	.7798	.8409	.9014	.9585	.9834	.9983	.9994
		2 .1803	.2236	.2737	.3314	.3982	.4757	.5667	.6756	.8109	.8944	.9755	.9873
		3 .0402	.0555	.0755	.1017	.1363	.1825	.2461	.3383	.4879	.6153	.8144	.8665
	4	1 .6263	.6800	.7324	.7831	.8318	.8780	.9206	.9579	.9868	.9961	.9998	1.0000
		2 .3399	.3984	.4615	.5292	.6015	.6786	.7602	.8455	.9316	.9716	.9969	.9988
		3 .1475	.1870	.2337	.2890	.3544	.4322	.5257	.6404	.7869	.8796	.9716	.9853
	5	1 .7603	.8034	.8434	.8802	.9133	.9424	.9667	.9852	.9966	.9993	1.0000	1.0000
		2 .5359	.5962	.6568	.7172	.7769	.8350	.8903	.9403	.9805	.9942	.9997	.9999
	6	1 .8859	.9114	.9337	.9528	.9687	.9814	.9907	.9967	.9995	.9999	1.0000	1.0000
8	1	1 .1559	.1908	.2303	.2752	.3265	.3861	.4571	.5458	.6685	.7602	.8894	.9212
	2	1 .2981	.3500	.4059	.4662	.5316	.6028	.6813	.7692	.8710	.9301	.9844	.9920
		2 .0498	.0679	.0910	.1207	.1588	.2085	.2752	.3690	.5169	.6396	.8275	.8762
	3	1 .4319	.4909	.5518	.6147	.6795	.7461	.8142	.8829	.9496	.9795	.9978	.9992
		2 .1399	.1790	.2255	.2806	.3460	.4240	.5180	.6337	.7821	.8765	.9707	.9848
		3 .0264	.0380	.0540	.0757	.1054	.1467	.2059	.2950	.4463	.5798	.7947	.8519

Table 2.3 (continued)

n	s	k\ρ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
8	4	1	.5590	.6178	.6761	.7338	.7903	.8451	.8969	.9437	.9815	.9941	.9997	.9999
		2	.2644	.3205	.3831	.4524	.5288	.6129	.7049	.8048	.9103	.9616	.9956	.9983
		3	.0971	.1292	.1691	.2184	.2795	.3555	.4510	.5738	.7393	.8492	.9632	.9807
		4	.0215	.0315	.0455	.0650	.0922	.1309	.1873	.2742	.4254	.5615	.7843	.8442
	5	1	.6798	.7324	.7825	.8297	.8736	.9134	.9480	.9758	.9941	.9987	1.0000	1.0000
		2	.4185	.4830	.5506	.6207	.6928	.7660	.8387	.9081	.9680	.9899	.9995	.9999
		3	.2250	.2784	.3392	.4080	.4857	.5729	.6705	.7788	.8963	.9550	.9947	.9980
	6	1	.7945	.8353	.8724	.9057	.9347	.9590	.9782	.9915	.9985	.9998	1.0000	1.0000
		2	.5972	.6579	.7173	.7750	.8300	.8812	.9270	.9650	.9911	.9980	1.0000	1.0000
	7	1	.9021	.9257	.9459	.9628	.9765	.9868	.9939	.9981	.9998	1.0000	1.0000	1.0000
9	1	1	.1407	.1746	.2133	.2576	.3087	.3686	.4404	.5306	.6564	.7510	.8848	.9179
	2	1	.2692	.3205	.3764	.4373	.5039	.5773	.6589	.7513	.8597	.9234	.9827	.9911
		2	.0399	.0558	.0776	.1038	.1395	.1870	.2519	.3450	.4948	.6212	.8176	.8689
	3	1	.3903	.4501	.5127	.5781	.6463	.7173	.7910	.8665	.9414	.9758	.9973	.9990
		2	.1121	.1475	.1905	.2428	.3063	.3837	.4791	.5992	.7576	.8609	.9664	.9824
		3	.0183	.0275	.0405	.0587	.0846	.1219	.1768	.2626	.4137	.5512	.7783	.8398
	4	1	.5055	.5674	.6298	.6924	.7548	.8162	.8755	.9305	.9764	.9927	.9996	.9999
		2	.2123	.2650	.3254	.3942	.4720	.5600	.6589	.7696	.8909	.9522	.9943	.9978
		3	.0677	.0939	.1279	.1716	.2277	.3002	.3948	.5214	.6998	.8231	.9557	.9766
		4	.0126	.0196	.0297	.0446	.0664	.0990	.1489	.2299	.3792	.5201	.7600	.8261
	5	1	.6156	.6743	.7313	.7862	.8383	.8868	.9301	.9662	.9913	.9980	1.0000	1.0000
		2	.3368	.4013	.4709	.5455	.6247	.7077	.7932	.8782	.9954	.9853	.9992	.9998
		3	.1575	.2040	.2594	.3250	.4023	.4929	.5989	.7223	.8640	.9389	.9924	.9971
		4	.0574	.0810	.1121	.1530	.2064	.2767	.3702	.4977	.6812	.8105	.9519	.9746
	6	1	.7207	.7715	.8190	.8627	.9021	.9364	.9646	.9854	.9972	.9995	1.0000	1.0000
		2	.4824	.5500	.6190	.6886	.7577	.8248	.8879	.9432	.9843	.9962	.9999	1.0000
		3	.2945	.3575	.4270	.5030	.5853	.6734	.7659	.8599	.9475	.9824	.9990	.9997
	7	1	.8206	.8592	.8937	.9238	.9493	.9699	.9852	.9949	.9993	.9999	1.0000	1.0000
		2	.6452	.7051	.7626	.8168	.8670	.9118	.9497	.9786	.9957	.9993	1.0000	1.0000
	8	1	.9145	.9365	.9549	.9700	.9817	.9903	.9959	.9989	.9999	1.0000	1.0000	1.0000
10	1	1	.1284	.1613	.1992	.2430	.2938	.3537	.4261	.5176	.6459	.7429	.8808	.9150
	2	1	.2457	.2963	.3519	.4131	.4806	.5555	.6396	.7358	.8498	.9175	.9812	.9903
		2	.0327	.0469	.0658	.0910	.1246	.1700	.2332	.3253	.4763	.6056	.8091	.8626
	3	1	.3565	.4165	.4800	.5472	.6180	.6925	.7706	.8519	.9339	.9723	.9969	.9988
		2	.0921	.1242	.1642	.2137	.2750	.3512	.4470	.5702	.7364	.8471	.9625	.9803
		3	.0133	.0207	.0315	.0471	.0700	.1037	.1550	.2374	.3875	.5277	.7645	.8295
	4	1	.4620	.5257	.5909	.6571	.7239	.7907	.8563	.9183	.9715	.9909	.9995	.9999
		2	.1747	.2239	.2815	.3487	.4266	.5165	.6201	.7390	.8733	.9434	.9930	.9973
		3	.0493	.0710	.1001	.1389	.1903	.2588	.3512	.4792	.6664	.8003	.9488	.9728
		4	.0080	.0129	.0206	.0322	.0501	.0778	.1222	.1977	.3436	.4871	.7399	.8109

Table 2.3 (continued)

n	s	k\p	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
10	5	1	.5631	.6258	.6878	.7484	.8070	.8625	.9134	.9568	.9883	.9972	.9999	1.0000
		2	.2777	.3401	.4094	.4857	.5688	.6583	.7532	.8507	.9431	.9806	.9988	.9997
		3	.1151	.1551	.2047	.2657	.3402	.4309	.5409	.6744	.8348	.9237	.9901	.9962
		4	.0364	.0540	.0785	.1122	.1585	.2223	.3113	.4390	.6334	.7772	.9416	.9689
		5	.0067	.0111	.0179	.0284	.0449	.0708	.1131	.1861	.3303	.4744	.7320	.8049
	6	1	.6600	.7177	.7727	.8244	.8721	.9148	.9511	.9789	.9957	.9992	1.0000	1.0000
		2	.3986	.4682	.5415	.6178	.6960	.7747	.8513	.9215	.9769	.9941	.9998	1.0000
		3	.2160	.2736	.3402	.4162	.5021	.5979	.7032	.8156	.9270	.9744	.9984	.9996
		4	.1003	.1373	.1814	.2427	.3154	.4054	.5165	.6536	.8218	.9168	.9890	.9958
	7	1	.7528	.8018	.8466	.8870	.9224	.9520	.9752	.9908	.9986	.9998	1.0000	1.0000
		2	.5349	.6038	.6725	.7401	.8051	.8658	.9199	.9637	.9920	.9985	1.0000	1.0000
		3	.3554	.4248	.4994	.5785	.6612	.7459	.8300	.9087	.9725	.9929	.9998	1.0000
	8	1	.8411	.8778	.9098	.9372	.9597	.9772	.9896	.9968	.9997	1.0000	1.0000	1.0000
		2	.6837	.7423	.7973	.8481	.8937	.9329	.9644	.9864	.9978	.9997	1.0000	1.0000
	9	1	.9243	.9448	.9618	.9753	.9855	.9927	.9971	.9993	.9999	1.0000	1.0000	1.0000
11	1	1	.1182	.1502	.1873	.2305	.2810	.3409	.4137	.5062	.6368	.7359	.8773	.9125
	2	1	.2263	.2760	.3312	.3924	.4605	.5366	.6227	.7220	.8408	.9121	.9799	.9896
		2	.0274	.0401	.0574	.0808	.1126	.1562	.2177	.3088	.4604	.5922	.8016	.8571
	3	1	.3284	.3883	.4523	.5207	.5934	.6707	.7526	.8388	.9271	.9691	.9965	.9987
		2	.0773	.1065	.1437	.1906	.2498	.3245	.4202	.5454	.7179	.8349	.9590	.9784
		3	.0100	.0161	.0252	.0388	.0591	.0899	.1380	.2173	.3658	.5078	.7526	.8205
	4	1	.4259	.4907	.5576	.6265	.6969	.7681	.8389	.9070	.9667	.9892	.9994	.9998
		2	.1467	.1924	.2472	.3124	.3894	.4802	.5870	.7122	.8574	.9352	.9918	.9969
		3	.0372	.0554	.0805	.1150	.1622	.2268	.3165	.4443	.6377	.7802	.9425	.9693
		4	.0053	.0090	.0149	.0242	.0391	.0630	.1029	.1732	.3152	.4599	.7228	.7980
	5	1	.5194	.5849	.6504	.7154	.7791	.8405	.8977	.9478	.9854	.9964	.9999	1.0000
		2	.2335	.2930	.3608	.4371	.5223	.6162	.7181	.8258	.9314	.9760	.9985	.9996
		3	.0870	.1214	.1656	.2218	.2927	.3817	.4933	.6333	.8086	.9095	.9878	.9953
		4	.0243	.0378	.0573	.0854	.1255	.1832	.2670	.3926	.5933	.7482	.9322	.9636
		5	.0039	.0068	.0115	.0192	.0319	.0530	.0891	.1549	.2930	.4381	.7087	.7872
	6	1	.6094	.6719	.7324	.7903	.8447	.8945	.9378	.9722	.9940	.9989	1.0000	1.0000
		2	.3357	.4047	.4794	.5593	.6434	.7304	.8178	.9007	.9694	.9919	.9997	1.0000
		3	.1636	.2153	.2772	.3507	.4366	.5361	.6495	.7756	.9072	.9662	.9977	.9994
		4	.0673	.0967	.1358	.1870	.2536	.3399	.4515	.5962	.7841	.8960	.9856	.9943
		5	.0211	.0333	.0512	.0774	.1153	.1706	.2523	.3767	.5791	.7377	.9288	.9616
	7	1	.6958	.7521	.8048	.8532	.8968	.9344	.9647	.9863	.9978	.9997	1.0000	1.0000
		2	.4514	.5240	.5988	.6747	.7501	.8231	.8907	.9480	.9877	.9976	1.0000	1.0000
		3	.2702	.3363	.4105	.4927	.5822	.6778	.7771	.8750	.9598	.9889	.9996	.9999
		4	.1455	.1943	.2539	.3257	.4110	.5113	.6275	.7590	.8987	.9627	.9974	.9993
	8	1	.7787	.8257	.8681	.9054	.9372	.9630	.9821	.9941	.9993	.9999	1.0000	1.0000
		2	.5787	.6477	.7152	.7800	.8406	.8952	.9414	.9761	.9958	.9994	1.0000	1.0000
		3	.4084	.4820	.5592	.6389	.7196	.7991	.8740	.9390	.9851	.9970	1.0000	1.0000

Table 2.3 (continued)

n	s	k\ρ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
11	9	1	.8577	.8925	.9224	.9474	.9674	.9824	.9925	.9979	.9998	1.0000	1.0000	1.0000
		2	.7152	.7721	.8246	.8721	.9135	.9478	.9741	.9912	.9989	.9999	1.0000	1.0000
	10	1	.9322	.9515	.9671	.9793	.9883	.9944	.9979	.9996	1.0000	1.0000	1.0000	1.0000
12	1	1	.1097	.1407	.1771	.2198	.2700	.3298	.4028	.4962	.6286	.7296	.8742	.9102
	2	1	.2100	.2587	.3134	.3745	.4429	.5200	.6078	.7097	.8328	.9072	.9786	.9889
		2	.0234	.0349	.0508	.0727	.1028	.1448	.2047	.2947	.4467	.5804	.7950	.8522
	3	1	.3048	.3643	.4285	.4976	.5718	.6514	.7364	.8268	.9207	.9661	.9961	.9985
		2	.0659	.0927	.1274	.1719	.2289	.3022	.3973	.5239	.7014	.8239	.9557	.9766
		3	.0078	.0128	.0206	.0325	.0508	.0792	.1244	.2007	.3474	.4908	.7422	.8126
	4	1	.3954	.4607	.5289	.5998	.6730	.7478	.8231	.8965	.9623	.9876	.9993	.9998
		2	.1252	.1676	.2196	.2826	.3585	.4494	.5582	.6884	.8428	.9276	.9906	.9964
		3	.0288	.0443	.0662	.0971	.1405	.2014	.2881	.4151	.6129	.7624	.9368	.9661
		4	.0037	.0065	.0111	.0188	.0314	.0523	.0883	.1540	.2921	.4372	.7080	.7867
	5	1	.4825	.5497	.6178	.6862	.7542	.8204	.8832	.9393	.9824	.9956	.9999	1.0000
		2	.1994	.2559	.3216	.3971	.4831	.5798	.6871	.8031	.9204	.9715	.9981	.9995
		3	.0675	.0973	.1368	.1885	.2555	.3420	.4536	.5979	.7849	.8963	.9856	.9943
		4	.0169	.0274	.0432	.0668	.1019	.1540	.2327	.3551	.5592	.7228	.9236	.9587
		5	.0024	.0044	.0078	.0136	.0236	.0410	.0722	.1318	.2637	.4084	.6888	.7719
	6	1	.5664	.6323	.6970	.7598	.8198	.8756	.9252	.9656	.9922	.9985	1.0000	1.0000
		2	.2870	.3543	.4288	.5104	.5983	.6914	.7872	.8810	.9618	.9895	.9997	.9999
		3	.1273	.1732	.2303	.3000	.3843	.4849	.6034	.7399	.8884	.9581	.9970	.9992
		4	.0470	.0706	.1032	.1479	.2083	.2898	.3995	.5480	.7503	.8765	.9822	.9929
		5	.0130	.0217	.0350	.0556	.0869	.1348	.2091	.3284	.5340	.7035	.9169	.9549
		6	.0021	.0038	.0069	.0123	.0215	.0378	.0675	.1252	.2550	.3994	.6825	.7671
	7	1	.6473	.7089	.7675	.8225	.8728	.9173	.9543	.9816	.9968	.9995	1.0000	1.0000
		2	.3866	.4601	.5379	.6189	.7018	.7842	.8629	.9324	.9830	.9965	.9999	1.0000
		3	.2107	.2719	.3432	.4250	.5173	.6195	.7297	.8431	.9467	.9847	.9995	.9999
		4	.1020	.1427	.1948	.2605	.3422	.4426	.5642	.7087	.8716	.9507	.9964	.9990
		5	.0417	.0634	.0940	.1364	.1947	.2742	.3829	.5321	.7389	.8698	.9810	.9924
	8	1	.7252	.7798	.8301	.8755	.9152	.9484	.9740	.9909	.9988	.9999	1.0000	1.0000
		2	.4967	.5709	.6458	.7201	.7919	.8589	.9181	.9648	.9932	.9990	1.0000	1.0000
		3	.3195	.3919	.4712	.5565	.6465	.7389	.8302	.9137	.9773	.9951	.9999	1.0000
		4	.1902	.2489	.3185	.3995	.4923	.5967	.7108	.8302	.9413	.9829	.9994	.9999
	9	1	.8000	.8451	.8851	.9196	.9483	.9707	.9868	.9961	.9996	1.0000	1.0000	1.0000
		2	.6156	.6840	.7498	.8115	.8677	.9167	.9563	.9839	.9977	.9997	1.0000	1.0000
		3	.4546	.5307	.6089	.6877	.7652	.8389	.9050	.9584	.9918	.9987	1.0000	1.0000
	10	1	.8714	.9044	.9323	.9553	.9732	.9862	.9945	.9986	.9999	1.0000	1.0000	1.0000
		2	.7414	.7965	.8466	.8908	.9285	.9587	.9808	.9941	.9994	1.0000	1.0000	1.0000
	11	1	.9387	.9568	.9713	.9824	.9904	.9956	.9985	.9997	1.0000	1.0000	1.0000	1.0000
13	1	1	.1023	.1326	.1683	.2104	.2602	.3199	.3932	.4873	.6213	.7239	.8714	.9081
	2	1	.1960	.2439	.2979	.3588	.4275	.5053	.5944	.6986	.8255	.9028	.9774	.9883
		2	.0202	.0307	.0454	.0660	.0947	.1351	.1935	.2825	.4347	.5699	.7891	.8478

Table 2.3 (continued)

n	s	$k \backslash \rho$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
13	3	1	.2845	.3435	.4077	.4774	.5527	.6341	.7218	.8160	.9149	.9633	.9957	.9984
		2	.0569	.0816	.1141	.1564	.2115	.2831	.3776	.5050	.6867	.8139	.9528	.9750
		3	.0062	.0104	.0172	.0278	.0443	.0706	.1133	.1869	.3317	.4759	.7329	.8056
	4	1	.3693	.4347	.5038	.5762	.6516	.7295	.8087	.8868	.9580	.9859	.9992	.9998
		2	.1082	.1478	.1972	.2580	.3323	.4229	.5331	.6671	.8295	.9205	.9895	.9959
		3	.0229	.0361	.0554	.0833	.1233	.1808	.2646	.3901	.5910	.7464	.9315	.9632
		4	.0027	.0048	.0086	.0149	.0257	.0442	.0769	.1386	.2727	.4177	.6950	.7767
	5	1	.4508	.5192	.5892	.6603	.7316	.8021	.8697	.9312	.9796	.9947	.9998	1.0000
		2	.1726	.2260	.2894	.3636	.4496	.5481	.6594	.7823	.9099	.9671	.9978	.9994
		3	.0536	.0796	.1149	.1624	.2256	.3093	.4200	.5670	.7635	.8840	.9835	.9935
		4	.0122	.0205	.0335	.0536	.0843	.1316	.2054	.3241	.5299	.7002	.9157	.9542
		5	.0015	.0030	.0055	.0100	.0181	.0326	.0598	.1141	.2400	.3837	.6715	.7585
	6	1	.5295	.5978	.6657	.7325	.7971	.8580	.9132	.9592	.9904	.9981	1.0000	1.0000
		2	.2487	.3135	.3870	.4690	.5593	.6568	.7595	.8626	.9543	.9870	.9996	.9999
		3	.1013	.1421	.1943	.2600	.3417	.4420	.5634	.7077	.8707	.9502	.9963	.9989
		4	.0339	.0530	.0805	.1195	.1742	.2506	.3572	.5070	.7201	.8585	.9789	.9915
		5	.0084	.0147	.0249	.0413	.0673	.1090	.1765	.2900	.4961	.6735	.9060	.9486
		6	.0012	.0023	.0044	.0082	.0152	.0281	.0528	.1036	.2254	.3679	.6602	.7497
	7	1	.6055	.6710	.7343	.7945	.8506	.9011	.9441	.9767	.9957	.9994	1.0000	1.0000
		2	.3353	.4081	.4870	.5711	.6591	.7489	.8369	.9171	.9781	.9953	.9999	1.0000
		3	.1679	.2238	.2911	.3707	.4635	.5694	.6875	.8133	.9338	.9802	.9993	.9999
		4	.0738	.1078	.1530	.2124	.2892	.3873	.5110	.6643	.8461	.9389	.9953	.9986
		5	.0271	.0434	.0675	.1026	.1531	.2255	.3292	.4790	.6988	.8455	.9764	.9905
		6	.0075	.0132	.0226	.0378	.0625	.1023	.1679	.2795	.4852	.6648	.9028	.9467
	8	1	.6789	.7393	.7958	.8478	.8943	.9341	.9657	.9875	.9982	.9998	1.0000	1.0000
		2	.4314	.5079	.5871	.6677	.7479	.8249	.8952	.9530	.9903	.9984	1.0000	1.0000
		3	.2552	.3239	.4020	.4891	.5842	.6855	.7893	.8886	.9688	.9929	.9999	1.0000
		4	.1381	.1891	.2522	.3291	.4211	.5291	.6526	.7883	.9226	.9763	.9991	.9998
		5	.0665	.0983	.1413	.1986	.2735	.3705	.4944	.6502	.8377	.9350	.9949	.9985
	9	1	.7497	.8026	.8506	.8931	.9294	.9588	.9804	.9938	.9993	.9999	1.0000	1.0000
		2	.5358	.6107	.6849	.7568	.8246	.8859	.9375	.9756	.9962	.9995	1.0000	1.0000
		3	.3641	.4410	.5234	.6098	.6984	.7862	.8688	.9393	.9869	.9978	1.0000	1.0000
		4	.2330	.2999	.3769	.4641	.5607	.6650	.7734	.8787	.9654	.9920	.9998	1.0000
	10	1	.8178	.8611	.8988	.9309	.9569	.9767	.9901	.9973	.9998	1.0000	1.0000	1.0000
		2	.6470	.7145	.7782	.8368	.8889	.9329	.9668	.9889	.9987	.9999	1.0000	1.0000
		3	.4950	.5725	.6506	.7276	.8013	.8691	.9272	.9710	.9953	.9994	1.0000	1.0000
	11	1	.8828	.9142	.9404	.9615	.9776	.9890	.9958	.9991	1.0000	1.0000	1.0000	1.0000
		2	.7635	.8168	.8644	.9058	.9401	.9669	.9855	.9960	.9997	1.0000	1.0000	1.0000
	12	1	.9442	.9613	.9747	.9849	.9920	.9965	.9989	.9998	1.0000	1.0000	1.0000	1.0000
14	1	1	.0960	.1255	.1605	.2021	.2516	.3111	.3845	.4792	.6147	.7188	.8688	.9063
	2	1	.1839	.2309	.2844	.3450	.4137	.4921	.5823	.6886	.8188	.8987	.9763	.9877
		2	.0176	.0273	.0410	.0604	.0878	.1268	.1839	.2717	.4239	.5605	.7837	.8438

Table 2.3 (continued)

n	s	k\ρ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
14	3	1	.2670	.3254	.3894	.4594	.5356	.6185	.7086	.8060	.9094	.9606	.9953	.9982
		2	.0498	.0726	.1031	.1435	.1966	.2667	.3603	.4883	.6734	.8048	.9500	.9734
		3	.0050	.0086	.0146	.0240	.0392	.0636	.1040	.1751	.3180	.4627	.7246	.7993
	4	1	.3466	.4120	.4816	.5552	.6324	.7128	.7954	.8778	.9540	.9844	.9991	.9997
		2	.0947	.1316	.1785	.2371	.3099	.3999	.5108	.6480	.8173	.9138	.9884	.9955
		3	.0185	.0299	.0471	.0724	.1094	.1638	.2447	.3686	.5716	.7319	.9266	.9604
		4	.0020	.0037	.0068	.0121	.0215	.0380	.0679	.1260	.2563	.4008	.6835	.7678
	5	1	.4233	.4924	.5638	.6371	.7112	.7853	.8572	.9234	.9768	.9939	.9998	1.0000
		2	.1510	.2015	.2625	.3351	.4207	.5202	.6346	.7633	.9000	.9628	.9974	.9993
		3	.0434	.0662	.0980	.1417	.2013	.2820	.3912	.5399	.7441	.8726	.9814	.9926
		4	.0090	.0157	.0266	.0438	.0710	.1141	.1833	.2982	.5043	.6801	.9084	.9500
		5	.0010	.0021	.0040	.0076	.0142	.0265	.0505	.1002	.2205	.3626	.6563	.7467
	6	1	.4974	.5674	.6378	.7078	.7762	.8417	.9018	.9529	.9885	.9977	1.0000	1.0000
		2	.2178	.2800	.3519	.4336	.5252	.6260	.7342	.8452	.9470	.9846	.9995	.9999
		3	.0820	.1184	.1662	.2280	.3066	.4056	.5286	.6788	.8541	.9425	.9956	.9987
		4	.0252	.0408	.0641	.0983	.1479	.2193	.3222	.4718	.6930	.8418	.9756	.9901
		5	.0057	.0104	.0183	.0315	.0534	.0899	.1513	.2590	.4637	.6471	.8960	.9428
		6	.0007	.0015	.0029	.0057	.0111	.0215	.0424	.0875	.2017	.3418	.6408	.7345
	7	1	.5691	.6376	.7045	.7691	.8300	.8858	.9342	.9718	.9946	.9992	1.0000	1.0000
		2	.2940	.3652	.4441	.5298	.6214	.7169	.8126	.9024	.9732	.9940	.9999	1.0000
		3	.1362	.1870	.2500	.3267	.4184	.5262	.6497	.7858	.9211	.9756	.9990	.9998
		4	.0550	.0833	.1226	.1761	.2477	.3423	.4660	.6251	.8222	.9274	.9941	.9983
		5	.0183	.0307	.0499	.0791	.1229	.1885	.2865	.4346	.6633	.8230	.9719	.9885
		6	.0045	.0084	.0152	.0267	.0464	.0798	.1376	.2413	.4445	.6311	.8898	.9391
		7	.0006	.0013	.0026	.0052	.0102	.0200	.0400	.0836	.1959	.3352	.6358	.7306
	8	1	.6385	.7032	.7648	.8222	.8745	.9202	.9574	.9839	.9976	.9997	1.0000	1.0000
		2	.3787	.4555	.5370	.6218	.7082	.7934	.8732	.9412	.9872	.9978	1.0000	1.0000
		3	.2074	.2716	.3469	.4335	.5311	.6382	.7517	.8644	.9600	.9905	.9998	1.0000
		4	.1031	.1469	.2034	.2751	.3641	.4726	.6018	.7498	.9042	.9694	.9987	.9997
		5	.0451	.0700	.1054	.1547	.2224	.3142	.4370	.5992	.8059	.9194	.9933	.9980
		6	.0165	.0279	.0459	.0735	.1156	.1792	.2754	.4227	.6535	.8167	.9706	.9880
	9	1	.7057	.7646	.8190	.8681	.9110	.9466	.9738	.9913	.9990	.9999	1.0000	1.0000
		2	.4710	.5492	.6287	.7080	.7848	.8563	.9187	.9668	.9944	.9993	1.0000	1.0000
		3	.2966	.3712	.4541	.5442	.6398	.7379	.8339	.9198	.9814	.9966	1.0000	1.0000
		4	.1743	.2342	.3063	.3915	.4900	.6009	.7215	.8447	.9527	.9884	.9998	1.0000
		5	.0939	.1355	.1899	.2597	.3474	.4557	.5863	.7378	.8983	.9672	.9986	.9997
	10	1	.7705	.8217	.8674	.9072	.9405	.9666	.9850	.9957	.9996	1.0000	1.0000	1.0000
		2	.5699	.6447	.7176	.7870	.8507	.9066	.9516	.9828	.9978	.9998	1.0000	1.0000
		3	.4043	.4843	.5684	.6546	.7407	.8231	.8973	.9566	.9923	.9990	1.0000	1.0000
		4	.2734	.3467	.4292	.5202	.6180	.7197	.8206	.9122	.9792	.9962	1.0000	1.0000
	11	1	.8329	.8744	.9102	.9400	.9637	.9811	.9924	.9982	.9999	1.0000	1.0000	1.0000
		2	.6741	.7403	.8018	.8574	.9057	.9452	.9744	.9922	.9993	1.0000	1.0000	1.0000
		3	.5305	.6085	.6858	.7605	.8303	.8924	.9434	.9795	.9973	.9997	1.0000	1.0000

Table 2.3 (continued)

n	s	k\ρ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
14	12	1	.8925	.9224	.9471	.9666	.9811	.9910	.9968	.9994	1.0000	1.0000	1.0000	1.0000
		2	.7824	.8339	.8792	.9179	.9493	.9730	.9888	.9972	.9998	1.0000	1.0000	1.0000
	13	1	.9488	.9650	.9776	.9869	.9932	.9971	.9991	.9999	1.0000	1.0000	1.0000	1.0000
15	1	1	.0905	.1192	.1537	.1947	.2438	.3032	.3767	.4719	.6086	.7141	.8664	.9045
	2	1	.1733	.2194	.2723	.3326	.4013	.4801	.5714	.6794	.8126	.8948	.9753	.9872
		2	.0156	.0244	.0373	.0556	.0818	.1196	.1755	.2622	.4143	.5520	.7788	.8401
	3	1	.2517	.3094	.3731	.4433	.5202	.6044	.6964	.7968	.9043	.9581	.9950	.9981
		2	.0439	.0651	.0939	.1324	.1838	.2524	.3450	.4733	.6613	.7964	.9474	.9720
		3	.0041	.0073	.0125	.0210	.0349	.0577	.0962	.1650	.3060	.4510	.7171	.7936
	4	1	.3268	.3920	.4618	.5363	.6150	.6976	.7831	.8694	.9501	.9829	.9989	.9997
		2	.0836	.1182	.1628	.2193	.2905	.3796	.4910	.6307	.8060	.9076	.9874	.9951
		3	.0152	.0252	.0405	.0636	.0980	.1496	.2278	.3498	.5542	.7188	.9221	.9578
		4	.0015	.0029	.0055	.0100	.0182	.0330	.0606	.1155	.2422	.3860	.6731	.7598
	5	1	.3993	.4687	.5412	.6161	.6927	.7698	.8445	.9161	.9740	.9930	.9998	1.0000
		2	.1335	.1812	.2398	.3107	.3954	.4955	.6123	.7459	.8907	.9588	.9970	.9992
		3	.0357	.0558	.0845	.1250	.1812	.2589	.3664	.5158	.7263	.8620	.9794	.9917
		4	.0069	.0123	.0215	.0364	.0606	.1001	.1651	.2762	.4819	.6619	.9016	.9460
		5	.0007	.0015	.0030	.0059	.0114	.0220	.0433	.0890	.2041	.3445	.6429	.7361
	6	1	.4693	.5405	.6128	.6853	.7571	.8265	.8911	.9468	.9867	.9972	1.0000	1.0000
		2	.1926	.2521	.3221	.4030	.4952	.5983	.7110	.8290	.9400	.9821	.9993	.9999
		3	.0675	.1000	.1438	.2018	.2773	.3745	.4981	.6527	.8386	.9351	.9949	.9985
		4	.0192	.0321	.0520	.0822	.1271	.1939	.2929	.4413	.6686	.8263	.9725	.9888
		5	.0040	.0075	.0138	.0246	.0432	.0753	.1314	.2334	.4357	.6235	.8867	.9373
		6	.0005	.0010	.0020	.0041	.0083	.0169	.0347	.0750	.1825	.3197	.6238	.7211
	7	1	.5372	.6078	.6777	.7458	.8109	.8714	.9246	.9670	.9934	.9989	1.0000	1.0000
		2	.2602	.3293	.4074	.4939	.5879	.6878	.7901	.8882	.9682	.9926	.9999	1.0000
		3	.1123	.1583	.2171	.2904	.3803	.4886	.6160	.7603	.9089	.9711	.9988	.9998
		4	.0419	.0657	.0999	.1480	.2145	.3052	.4274	.5902	.7999	.9163	.9929	.9979
		5	.0128	.0223	.0378	.0623	.1005	.1598	.2520	.3970	.6315	.8022	.9674	.9866
		6	.0029	.0056	.0106	.0195	.0354	.0636	.1147	.2110	.4101	.6014	.8777	.9320
		7	.0004	.0008	.0017	.0035	.0072	.0148	.0311	.0689	.1726	.3082	.6145	.7137
	8	1	.6030	.6710	.7366	.7986	.8559	.9068	.9492	.9802	.9968	.9996	1.0000	1.0000
		2	.3355	.4115	.4939	.5815	.6725	.7642	.8522	.9295	.9839	.9971	1.0000	1.0000
		3	.1712	.2306	.3204	.3872	.4855	.5963	.7173	.8414	.9512	.9879	.9997	1.0000
		4	.0786	.1163	.1667	.2328	.3178	.4250	.5572	.7145	.8862	.9624	.9983	.9997
		5	.0315	.0512	.0804	.1229	.1837	.2695	.3893	.5547	.7764	.9043	.9916	.9975
		6	.0104	.0187	.0322	.0542	.0893	.1450	.2337	.3764	.6135	.7901	.9648	.9854
		7	.0026	.0051	.0097	.0181	.0331	.0601	.1096	.2041	.4019	.5942	.8747	.9302

Table 2.3 (continued)

n	s	k\e	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995
15	9	1	.6668	.7304	.7900	.8447	.8934	.9347	.9670	.9886	.9986	.9999	1.0000	1.0000
		2	.4177	.4972	.5800	.6644	.7482	.8282	.9001	.9576	.9923	.9990	1.0000	1.0000
		3	.2452	.3162	.3976	.4888	.5885	.6942	.8010	.9004	.9755	.9953	1.0000	1.0000
		4	.1333	.1862	.2525	.3338	.4314	.5456	.6746	.8123	.9397	.9845	.9997	1.0000
		5	.0659	.0997	.1460	.2083	.2902	.3959	.5293	.6920	.8745	.9577	.9981	.9996
		6	.0287	.0471	.0748	.1155	.1745	.2586	.3773	.5432	.7686	.9002	.9911	.9973
	10	1	.7286	.7860	.8382	.8846	.9242	.9563	.9797	.9939	.9994	1.0000	1.0000	1.0000
		2	.5060	.5851	.6643	.7416	.8147	.8808	.9361	.9761	.9966	.9997	1.0000	1.0000
		3	.3349	.4141	.5002	.5917	.6861	.7799	.8678	.9414	.9887	.9984	1.0000	1.0000
		4	.2097	.2770	.3563	.4474	.5494	.6602	.7751	.8849	.9707	.9942	.9999	1.0000
		5	.1227	.1733	.2377	.3176	.4146	.5294	.6606	.8024	.9356	.9832	.9996	1.0000
	11	1	.7883	.8378	.8814	.9187	.9492	.9726	.9884	.9970	.9998	1.0000	1.0000	1.0000
		2	.5997	.6741	.7454	.8119	.8718	.9226	.9621	.9877	.9987	.9999	1.0000	1.0000
		3	.4405	.5227	.6074	.6925	.7753	.8523	.9187	.9685	.9953	.9995	1.0000	1.0000
		4	.3111	.3894	.4758	.5688	.6660	.7638	.8567	.9356	.9873	.9981	1.0000	1.0000
	12	1	.8458	.8857	.9196	.9474	.9690	.9845	.9941	.9987	.9999	1.0000	1.0000	1.0000
		2	.6977	.7624	.8217	.8744	.9191	.9547	.9800	.9945	.9996	1.0000	1.0000	1.0000
		3	.5618	.6398	.7159	.7880	.8538	.9106	.9555	.9852	.9984	.9999	1.0000	1.0000
	13	1	.9008	.9294	.9526	.9707	.9839	.9926	.9975	.9996	1.0000	1.0000	1.0000	1.0000
		2	.7987	.8484	.8916	.9278	.9566	.9778	.9913	.9980	.9999	1.0000	1.0000	1.0000
	14	1	.9527	.9681	.9799	.9885	.9942	.9977	.9993	.9999	1.0000	1.0000	1.0000	1.0000

Again, if we write $x = \sqrt{2}\,X$ and $y = \sqrt{2}\,Y$, then from the functions $\phi(x)$ and $\phi(y)$ we have the weight factors $\exp(-X^2)$ and $\exp(-Y^2)$ in the Gaussian quadrature formula.

2. Determination of the size of subset

Often ${}_n\Pi_{1:1}(\rho)$ is not large, even though ρ is very close to 1. Hence, we have to choose a subset of size s to ensure that it includes the largest object with probability at least P . In the case of the bivariate normal distribution, the selection probability can be expressed by

$$
\begin{aligned}
{}_n\Pi_{s:1}(\rho) &= \sum_{i=1}^{s} \Pr\{R_{n-i+1,n} = n\} \\
&= \sum_{i=1}^{s} \Pr\{R_{n,n} = n-i+1\} \\
&= \sum_{i=1}^{s} \Pr\{R_{1,n} = i\} .
\end{aligned}
$$

Hence, for a given value of ρ , we can determine the smallest subset size s for which ${}_n\Pi_{s:1}(\rho) \geq P$. Table 2.4 gives s for various n and ρ values.

Table 2.4 $s = \min_{j} \left\{ j \; : \; {}_{n}\Pi_{j:1}(\rho) \geq P \right\}$

P\ρ	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	.99	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	.99
					< n = 2 >											< n = 3 >						
0.50	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	1	1	1	1	1	1	1
0.60	2	2	2	1	1	1	1	1	1	1	1	2	2	2	2	2	2	1	1	1	1	1
0.70	2	2	2	2	2	1	1	1	1	1	1	2	2	2	2	2	2	2	1	1	1	1
0.80	2	2	2	2	2	2	2	2	1	1	1	3	3	3	2	2	2	2	2	2	1	1
0.90	2	2	2	2	2	2	2	2	2	2	1	3	3	3	3	3	3	2	2	2	2	1
0.95	2	2	2	2	2	2	2	2	2	2	1	3	3	3	3	3	3	3	3	2	2	2
0.99	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	2
					< n = 4 >											< n = 5 >						
0.50	2	2	2	2	2	1	1	1	1	1	1	3	3	2	2	2	2	1	1	1	1	1
0.60	3	3	2	2	2	2	2	1	1	1	1	3	3	3	2	2	2	2	1	1	1	1
0.70	3	3	3	3	2	2	2	2	1	1	1	4	4	3	3	3	2	2	2	1	1	1
0.80	4	3	3	3	3	3	2	2	2	1	1	4	4	4	4	3	3	3	2	2	1	1
0.90	4	4	4	4	3	3	3	3	2	2	1	5	5	5	4	4	4	3	3	2	2	1
0.95	4	4	4	4	4	4	3	3	3	2	2	5	5	5	5	5	4	4	4	3	2	2
0.99	4	4	4	4	4	4	4	4	3	3	2	5	5	5	5	5	5	5	5	4	3	2
					< n = 6 >											< n = 7 >						
0.50	3	3	3	2	2	2	1	1	1	1	1	4	3	3	2	2	2	2	1	1	1	1
0.60	4	3	3	3	2	2	2	2	1	1	1	4	4	4	3	3	2	2	2	1	1	1
0.70	4	4	4	3	3	3	2	2	1	1	1	5	5	4	4	3	3	2	2	2	1	1
0.80	5	5	4	4	4	3	3	2	2	2	1	6	5	5	5	4	4	3	3	2	2	1
0.90	6	6	5	5	5	4	4	3	3	2	1	7	6	6	6	5	5	4	3	3	2	2
0.95	6	6	6	6	5	5	5	4	3	3	2	7	7	7	6	6	6	5	4	3	3	2
0.99	6	6	6	6	6	6	6	5	4	4	3	7	7	7	7	7	7	6	6	5	4	3
					< n = 8 >											< n = 9 >						
0.50	4	4	3	3	2	2	2	1	1	1	1	4	4	3	3	2	2	2	1	1	1	1
0.60	5	4	4	3	3	2	2	2	1	1	1	5	5	4	4	3	3	2	2	1	1	1
0.70	6	5	5	4	4	3	3	2	2	1	1	6	6	5	5	4	3	3	2	2	1	1
0.80	7	6	6	5	5	4	3	3	2	2	1	7	7	6	6	5	4	4	3	2	2	1
0.90	7	7	7	6	6	5	5	4	3	2	2	8	8	8	7	6	6	5	4	3	2	2
0.95	8	8	8	7	7	6	6	5	4	3	2	9	9	8	8	8	7	6	5	4	3	2
0.99	8	8	8	8	8	8	7	6	5	4	3	9	9	9	9	9	8	8	7	5	4	3
					< n =10 >											< n =11 >						
0.50	5	4	4	3	3	2	2	1	1	1	1	5	5	4	3	3	2	2	1	1	1	1
0.60	6	5	5	4	3	3	2	2	1	1	1	6	6	5	4	4	3	2	2	1	1	1
0.70	7	6	6	5	4	4	3	2	2	1	1	8	7	6	5	5	4	3	2	2	1	1
0.80	8	7	7	6	5	5	4	3	2	2	1	9	8	7	7	6	5	4	3	2	2	1
0.90	9	9	8	8	7	6	5	4	3	2	2	10	10	9	8	8	7	6	4	3	2	2
0.95	10	10	9	9	8	7	6	5	4	3	2	11	10	10	10	9	8	7	6	4	3	2
0.99	10	10	10	10	10	9	9	7	6	4	3	11	11	11	11	11	10	9	8	6	5	3

Table 2.4 (continued)

P\e	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	.99	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	.99
						< n =12 >											< n =13 >					
0.50	6	5	4	4	3	2	2	2	1	1	1	6	5	4	4	3	2	2	2	1	1	1
0.60	7	6	5	5	4	3	2	2	1	1	1	7	7	6	5	4	3	3	2	1	1	1
0.70	8	7	7	6	5	4	3	2	2	1	1	9	8	7	6	5	4	3	3	2	1	1
0.80	10	9	8	7	6	5	4	3	2	2	1	10	9	9	8	7	5	4	3	2	2	1
0.90	11	10	10	9	8	7	6	5	3	2	2	12	11	11	10	9	7	6	5	3	2	2
0.95	12	11	11	10	10	9	7	6	4	3	2	13	12	12	11	10	9	8	6	4	3	2
0.99	12	12	12	12	11	11	10	8	6	5	3	13	13	13	13	12	12	10	9	6	5	3
						< n =14 >											< n =15 >					
0.50	7	6	5	4	3	3	2	2	1	1	1	7	6	5	4	3	3	2	2	1	1	1
0.60	8	7	6	5	4	3	3	2	1	1	1	8	7	6	5	4	3	3	2	1	1	1
0.70	9	8	7	6	5	4	3	3	2	1	1	10	9	8	7	6	5	4	3	2	1	1
0.80	11	10	9	8	7	6	5	3	2	2	1	12	11	10	9	7	6	5	4	2	2	1
0.90	13	12	11	10	9	8	6	5	3	3	2	13	13	12	11	10	8	7	5	3	3	2
0.95	14	13	13	12	11	10	8	6	4	3	2	14	14	13	13	12	10	9	7	4	3	2
0.99	14	14	14	14	13	12	11	9	7	5	3	15	15	15	15	14	13	12	10	7	5	3
						< n =16 >											< n =17 >					
0.50	7	6	5	4	3	3	2	2	1	1	1	8	7	5	4	4	3	2	2	1	1	1
0.60	9	8	7	6	5	4	3	2	1	1	1	9	8	7	6	5	4	3	2	2	1	1
0.70	11	10	8	7	6	5	4	3	2	1	1	11	10	9	7	6	5	4	3	2	1	1
0.80	12	11	10	9	8	6	5	4	2	2	1	13	12	11	9	8	7	5	4	2	2	1
0.90	14	14	13	12	10	9	7	5	4	3	2	15	14	13	12	11	9	7	5	4	3	2
0.95	15	15	14	13	12	11	9	7	5	3	2	16	16	15	14	13	11	9	7	5	3	2
0.99	16	16	16	16	15	14	12	10	7	5	3	17	17	17	16	16	15	13	11	7	5	3
						< n =18 >											< n =19 >					
0.50	8	7	6	5	4	3	2	2	1	1	1	9	7	6	5	4	3	2	2	1	1	1
0.60	10	9	7	6	5	4	3	2	2	1	1	10	9	8	6	5	4	3	2	2	1	1
0.70	12	11	9	8	6	5	4	3	2	1	1	12	11	10	8	7	5	4	3	2	2	1
0.80	14	13	11	10	8	7	5	4	3	2	1	15	13	12	10	9	7	5	4	3	2	1
0.90	16	15	14	13	11	9	8	6	4	3	2	17	16	15	13	12	10	8	6	4	3	2
0.95	17	17	16	15	13	12	10	7	5	3	2	18	18	17	16	14	12	10	8	5	4	2
0.99	18	18	18	17	17	15	14	11	8	5	3	19	19	19	18	17	16	14	11	8	6	3
						< n =20 >											< n =21 >					
0.50	9	8	6	5	4	3	2	2	1	1	1	9	8	6	5	4	3	2	2	1	1	1
0.60	11	9	8	7	5	4	3	2	2	1	1	12	10	8	7	5	4	3	2	2	1	1
0.70	13	12	10	8	7	5	4	3	2	2	1	14	12	10	9	7	6	4	3	2	2	1
0.80	15	14	12	11	9	7	6	4	3	2	1	16	15	13	11	9	7	6	4	3	2	1
0.90	18	17	16	14	12	10	8	6	4	3	2	19	18	16	15	13	11	8	6	4	3	2
0.95	19	18	18	16	15	13	10	8	5	4	2	20	19	18	17	15	13	11	8	5	4	2
0.99	20	20	20	19	18	17	15	12	8	6	3	21	21	21	20	19	18	15	12	8	6	3

Table 2.4 (continued)

P\ρ	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	.99	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	.99
					< n =22 >											< n =23 >						
0.50	10	8	7	5	4	3	2	2	1	1	1	10	9	7	5	4	3	2	2	1	1	1
0.60	12	10	9	7	6	4	3	2	2	1	1	13	11	9	7	6	4	3	2	2	1	1
0.70	14	13	11	9	7	6	4	3	2	2	1	15	13	11	9	8	6	4	3	2	2	1
0.80	17	15	14	12	10	8	6	4	3	2	1	18	16	14	12	10	8	6	4	3	2	1
0.90	19	18	17	15	13	11	9	6	4	3	2	20	19	18	16	14	11	9	6	4	3	2
0.95	21	20	19	18	16	14	11	8	5	4	2	22	21	20	18	17	14	11	8	5	4	2
0.99	22	22	22	21	20	18	16	13	8	6	3	23	23	22	22	21	19	16	13	8	6	3
					< n =24 >											< n =25 >						
0.50	11	9	7	6	4	3	2	2	1	1	1	11	9	7	6	4	3	3	2	1	1	1
0.60	13	11	9	7	6	4	3	2	2	1	1	14	12	10	8	6	5	3	2	2	1	1
0.70	16	14	12	10	8	6	4	3	2	2	1	16	14	12	10	8	6	5	3	2	2	1
0.80	18	17	15	13	10	8	6	4	3	2	1	19	17	15	13	11	8	6	4	3	2	1
0.90	21	20	18	16	14	12	9	6	4	3	2	22	21	19	17	15	12	9	7	4	3	2
0.95	23	22	21	19	17	15	12	9	5	4	2	24	23	22	20	18	15	12	9	5	4	2
0.99	24	24	23	23	22	20	17	13	9	6	3	25	25	24	24	22	20	18	14	9	6	3
					< n =26 >											< n =27 >						
0.50	11	9	8	6	5	3	3	2	1	1	1	12	10	8	6	5	4	3	2	1	1	1
0.60	14	12	10	8	6	5	3	2	2	1	1	15	12	10	8	6	5	3	2	2	1	1
0.70	17	15	13	10	8	6	5	3	2	2	1	17	15	13	11	8	6	5	3	2	2	1
0.80	20	18	16	13	11	9	6	4	3	2	1	20	18	16	14	11	9	7	5	3	2	1
0.90	23	21	20	18	15	12	10	7	4	3	2	24	22	20	18	16	13	10	7	4	3	2
0.95	25	24	22	21	18	16	12	9	6	4	2	25	25	23	21	19	16	13	9	6	4	2
0.99	26	26	25	24	23	21	18	14	9	6	3	27	27	26	25	24	22	19	14	9	6	3
					< n =28 >											< n =29 >						
0.50	12	10	8	6	5	4	3	2	1	1	1	13	10	8	6	5	4	3	2	1	1	1
0.60	15	13	11	8	7	5	4	2	2	1	1	16	13	11	9	7	5	4	3	2	1	1
0.70	18	16	13	11	9	7	5	3	2	2	1	19	16	14	11	9	7	5	3	2	2	1
0.80	21	19	17	14	12	9	7	5	3	2	1	22	20	17	15	12	9	7	5	3	2	1
0.90	25	23	21	19	16	13	10	7	4	3	2	25	24	22	19	17	13	10	7	4	3	2
0.95	26	25	24	22	20	17	13	9	6	4	2	27	26	25	23	20	17	13	10	6	4	2
0.99	28	28	27	26	25	22	19	15	9	6	3	29	29	28	27	26	23	20	15	9	6	3
					< n =30 >																	
0.50	13	11	9	7	5	4	3	2	1	1	1											
0.60	16	14	11	9	7	5	4	3	2	1	1											
0.70	19	17	14	12	9	7	5	3	2	2	1											
0.80	23	20	18	15	12	10	7	5	3	2	1											
0.90	26	25	23	20	17	14	10	7	4	3	2											
0.95	28	27	26	24	21	18	14	10	6	4	2											
0.99	30	30	29	28	26	24	20	15	10	6	3											

III. APPROXIMATE PROBABILITY WHEN THE CORRELATION ρ IS CLOSE TO 1 IN THE BIVARIATE NORMAL CASE

A. Limiting Probability When ρ Approaches 1

Let (X,Y) have the bivariate normal distribution with $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$, and correlation ρ . Then the p.d.f. of (X,Y) is defined as

$$f_{X,Y}(x,y;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp[-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}] ,$$

$$-\infty < x,y < \infty . \tag{3.1}$$

This standard bivariate normal p.d.f. has infinite value on the line $x = y$ when ρ goes to 1. Hence, this singularity makes computing $d\,{}_n\Pi_{s:k}(\rho)/d\rho$ difficult at the point $\rho = 1$. We can expect that the limit value of ${}_n\Pi_{s:k}(\rho)$, when ρ goes to 1, is 1. But, we will give the theoretical proof of this fact in this section. The p.d.f. $f(x,y;\rho) = f(x,y)$ in (3.1) can be expressed by the product form of the function $\phi(\cdot)$, i.e.,

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp[-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}]$$

$$= \frac{1}{\sqrt{1-\rho^2}} \phi(y)\phi(\frac{x-\rho y}{\sqrt{1-\rho^2}})$$

or

$$= \frac{1}{\sqrt{1-\rho^2}} \phi(x)\phi(\frac{y-\rho x}{\sqrt{1-\rho^2}}) . \tag{3.2}$$

Using this form, we can prove the next four lemmas which are needed in proving that $\lim\limits_{\rho\to 1} {}_n\Pi_{s:k}(\rho) = 1$.

Lemma 3.1: Let g^-, g^+, h^-, and h^+ be the functions defined in (2.9). Then we can express them in the following simple forms:

(i) $g^-(x,y) = \phi(y)\Phi(\frac{x-\rho y}{\sqrt{1-\rho^2}}) = \phi(y)[1 - \Phi(\frac{\rho y-x}{\sqrt{1-\rho^2}})]$

(ii) $g^+(x,y) = \phi(y)\Phi(\frac{\rho y-x}{\sqrt{1-\rho^2}}) = \phi(y)[1 - \Phi(\frac{x-\rho y}{\sqrt{1-\rho^2}})]$

(iii) $h^-(x,y) = \phi(x)\Phi(\frac{y-\rho x}{\sqrt{1-\rho^2}}) = \phi(x)[1 - \Phi(\frac{\rho x-y}{\sqrt{1-\rho^2}})]$

(iv) $h^+(x,y) = \phi(x)\Phi(\frac{\rho x-y}{\sqrt{1-\rho^2}}) = \phi(x)[1 - \Phi(\frac{y-\rho x}{\sqrt{1-\rho^2}})]$.

Proof: By definition (2.9) we have, using (3.2),

$$g^-(x,y) = \int_{-\infty}^{x} f(u,y)du$$

$$= \frac{1}{\sqrt{1-\rho^2}}\,\phi(y)\int_{-\infty}^{x}\phi(\frac{u-\rho y}{\sqrt{1-\rho^2}})\,du$$

$$= \phi(y)\int_{-\infty}^{\frac{x-\rho y}{\sqrt{1-\rho^2}}}\phi(z)dz \quad (z = \frac{u-\rho y}{\sqrt{1-\rho^2}})$$

$$= \phi(y)\Phi(\frac{x-\rho y}{\sqrt{1-\rho^2}}) .$$

The second equality holds since $\Phi(z) + \Phi(-z) = 1$. The other three can be proved similarly. Q.E.D.

Lemma 3.2: Let θ_i be defined by (1.3) with the standard bivariate normal p.d.f. (3.1). Then, we have

(i) $\theta_1(x,y) = \int_{-\infty}^{y} \phi(v)\Phi(\frac{x-\rho v}{\sqrt{1-\rho^2}})\ dv = \int_{-\infty}^{x} \phi(v)\Phi(\frac{y-\rho v}{\sqrt{1-\rho^2}})\ dv$

(ii) $\theta_2(x,y) = \int_{-\infty}^{x} \phi(v)\Phi(\frac{\rho v-y}{\sqrt{1-\rho^2}})\ dv$

(iii) $\theta_3(x,y) = \int_{-\infty}^{y} \phi(v)\Phi(\frac{\rho v-x}{\sqrt{1-\rho^2}})\ dv = \theta_2(y,x)$

(iv) $\theta_4(x,y) = \int_{x}^{\infty} \phi(v)\Phi(\frac{\rho v-y}{\sqrt{1-\rho^2}})\ dv = \int_{y}^{\infty} \phi(v)\Phi(\frac{\rho v-x}{\sqrt{1-\rho^2}})\ dv$.

Proof: (i) By definition of $\theta_1(x,y)$, we have

$$\theta_1(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$$

$$= \int_{-\infty}^{y} g^-(x,v)\ dv$$

$$= \int_{-\infty}^{y} \phi(v)\Phi(\frac{x-\rho v}{\sqrt{1-\rho^2}})\ dv \quad \text{(by Lemma 3.1 (i))} .$$

Also,

$$\theta_1(x,y) = \int_{-\infty}^{x} h^-(v,y)\ dv$$

$$= \int_{-\infty}^{x} \phi(v)\Phi(\frac{y-\rho v}{\sqrt{1-\rho^2}})\ dv \quad \text{(by Lemma 3.1 (iii))} .$$

Similarly, we can prove (ii), (iii), and (iv). Q.E.D.

<u>Lemma 3.3:</u> When ρ goes to 1 , the following limiting equalities hold.

$$\text{(i)} \quad \lim_{\rho\to 1} \theta_1(x,y) = \begin{cases} \Phi(x) & \text{if } y \geq x \\ \Phi(y) & \text{if } y < x \end{cases}$$

$$\text{(ii)} \quad \lim_{\rho\to 1} \theta_2(x,y) = \begin{cases} 0 & \text{if } y \geq x \\ \Phi(x) - \Phi(y) & \text{if } y < x \end{cases}$$

$$\text{(iii)} \quad \lim_{\rho\to 1} \theta_3(x,y) = \begin{cases} \Phi(y) - \Phi(x) & \text{if } y \geq x \\ 0 & \text{if } y < x \end{cases}$$

$$\text{(iv)} \quad \lim_{\rho\to 1} \theta_4(x,y) = \begin{cases} 1 - \Phi(y) & \text{if } y \geq x \\ 1 - \Phi(x) & \text{if } y < x \end{cases}$$

<u>Proof:</u> In the second expression for θ_1 in Lemma 3.2 (i), we see that the integrand is bounded by $\phi(v)$ which is obviously integrable on $(-\infty,x)$. Also, in this range of v , we have for $x \leq y$,

$$\lim_{\rho\to 1} \Phi\left(\frac{y-\rho v}{\sqrt{1-\rho^2}}\right) = \Phi(\infty) = 1 .$$

Hence, by the bounded convergence theorem

$$\lim_{\rho\to 1} \theta_1(x,y) = \int_{-\infty}^{x} \phi(v)\, dv = \Phi(x) .$$

Similarly for $y < x$ we have, from the first equality in Lemma 3.2 (i),

$$\lim_{\rho\to 1} \theta_1(x,y) = \int_{-\infty}^{y} \phi(v) \lim_{\rho\to 1}\left(\frac{x-\rho v}{\sqrt{1-\rho^2}}\right) dv$$

$$= \int_{-\infty}^{y} \phi(v)\, dv = \Phi(y) .$$

By the same arguments, (ii), (iii), and (iv) can be proved. Q.E.D.

<u>Lemma 3.4:</u> Let g^-, g^+, h^-, and h^+ be defined in (2.9) with the standard bivariate normal density. Then

$$\text{(i)}\quad \lim_{\rho\to 1} g^-(x,y) = \begin{cases} 0 & \text{if } y > x \\ \tfrac{1}{2}\phi(y) & \text{if } y = x \\ \phi(y) & \text{if } y < x \end{cases}$$

$$\text{(ii)}\quad \lim_{\rho\to 1} g^+(x,y) = \begin{cases} \phi(y) & \text{if } y > x \\ \tfrac{1}{2}\phi(y) & \text{if } y = x \\ 0 & \text{if } y < x \end{cases}$$

$$\text{(iii)}\quad \lim_{\rho\to 1} h^-(x,y) = \begin{cases} \phi(x) & \text{if } y > x \\ \tfrac{1}{2}\phi(x) & \text{if } y = x \\ 0 & \text{if } y < x \end{cases}$$

$$\text{(iv)}\quad \lim_{\rho\to 1} h^+(x,y) = \begin{cases} 0 & \text{if } y > x \\ \tfrac{1}{2}\phi(x) & \text{if } y = x \\ \phi(x) & \text{if } y < x \end{cases} .$$

<u>Proof:</u> When $x = y$, we have from Lemma 3.1

$$\begin{aligned} \lim_{\rho\to 1} g^-(x,y) &= \phi(y) \lim_{\rho\to 1} \Phi\left(\frac{y(1-\rho)}{\sqrt{1-\rho^2}}\right) \\ &= \phi(y) \lim_{\rho\to 1} \Phi\left(\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\, y\right) \\ &= \tfrac{1}{2}\phi(y) . \end{aligned}$$

For $y > x$ $\lim_{\rho\to 1} \Phi[(x-\rho y)/\sqrt{1-\rho^2}] = \Phi(-\infty) = 0$ and for $y < x$ $\lim_{\rho\to 1} \Phi[(x-\rho y)/\sqrt{1-\rho^2}] = \Phi(\infty) = 1$. Hence, we have the result of (i).

Similarly (ii), (iii), and (iv) can be proved. Q.E.D.

From Lemma 2.1 we have, for $1 \le k \le s < n$,

$$
\begin{aligned}
{}_n\Pi_{s:k} &= {}_nP^{(3)}_{s+1:k} \\
&= \frac{n!}{(n-s-1)!\,(s-k)!\,(k-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1^{n-s-1}\theta_3^{s-k}\theta_4^{k-1}g^{+}h^{-}dxdy \; .
\end{aligned}
$$

The integrand and the integral itself are bounded. Hence, by the bounded convergence theorem we have

$$
\lim_{\rho\to 1} {}_n\Pi_{s:k}(\rho) = \frac{n!}{(n-s-1)!\,(s-k)!\,(k-1)!}\iint\limits_{y>x} \Phi^{n-s-1}_{(x)}[\Phi(y)-\Phi(x)]^{s-k}[1-\Phi(y)]^{k-1}
$$

$$
\cdot\ \phi(x)\phi(y)dxdy \; , \qquad (3.3)
$$

since, for $s \neq k$, $\lim_{\rho\to 1}\theta_3(x,x) = 0$ and for $s = k$, the measure on the line $y = x$ is zero with respect to $\phi(x)\phi(y)dxdy$. The integrand of (3.3) is the joint p.d.f. of order statistics $Z_{(n-s)}$ and $Z_{(n-k+1)}$ from the standard normal sample of size n . See David (1981). Hence, we have

$$
\lim_{\rho\to 1} {}_n\Pi_{s:k}(\rho) = 1 \quad \text{for} \quad 1 \le k \le s < n \; . \qquad (3.4)
$$

Also, note that

$$
\left.
\begin{aligned}
&\lim_{\rho\to 1} {}_nP^{(1)}_{s:s} = 1 \; , \; \lim_{\rho\to 1} {}_nP^{(3)}_{s:s} = 0 \; , \\
&\lim_{\rho\to 1} {}_nP^{(1)}_{s:k} = 0 \ (s > k) \; , \; \lim_{\rho\to 1} {}_nP^{(3)}_{s:k} = 1 \ (s > k) \; , \\
&\lim_{\rho\to 1} {}_nP^{(2)}_{s:k} = 0 \text{ for all } k \; .
\end{aligned}
\right\} \qquad (3.5)
$$

B. Derivatives of ${}_n\Pi_{s:k}(\rho)$

Sheppard (1900) proved that for $x > 0$ and $y > 0$,

$$\theta_4(x,y) = \int_y^\infty \int_x^\infty f(u,v;\rho)dudv$$

$$= \frac{1}{2\pi} \int_{\cos^{-1}\rho}^{\pi} \exp[-\tfrac{1}{2}(x^2-2xy\cos w+y^2)\operatorname{cosec}^2 w]\ dw\ . \qquad (3.6)$$

From this equation, we have that for $x > 0$ and $y > 0$,

$$\frac{d}{d\rho}\,\theta_4(x,y) = (\frac{d}{d\rho}\cos^{-1}\rho)\ \frac{-1}{2\pi}\exp[-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}]$$

$$= (\frac{-1}{\sqrt{1-\rho^2}})\ (\frac{-1}{2\pi})\ \exp[-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}]$$

$$= f(x,y;\rho)\ . \qquad (3.7)$$

It can easily be seen that (3.7) continues to hold for any x and y . By the definition of the θ_i, we have that for any x and y ,

$$\frac{d}{d\rho}\,\theta_1(x,y) = \frac{d}{d\rho}\,\theta_4(x,y) = f(x,y;\rho)\ . \qquad (3.8)$$

$$\frac{d}{d\rho}\,\theta_2(x,y) = \frac{d}{d\rho}\,\theta_3(x,y) = -f(x,y;\rho)\ . \qquad (3.9)$$

Using Lemma 2.1, we will now obtain the derivative of ${}_n\Pi_{s:k}(\rho)$ exactly for $-1 < \rho < 1$. Since the integrand of (2.20) is continuous and has a continuous derivative with respect to ρ for $-1 < \rho < 1$, we may differentiate the integral with respect to ρ under the integral sign. See Courant and John (1974). For $1 \leq k \leq s < n$, we have

$$\frac{d}{d\rho}\, {}_n\Pi_{s:k}(\rho) = \frac{d}{d\rho}\left\{\frac{n!}{(n-s-1)!(s-k)!(k-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1^{n-s-1}\theta_3^{s-k}\theta_4^{k-1}g^+h^-dxdy\right\}$$

$$= \frac{n!}{(n-s-2)!(s-k)!(k-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1^{n-s-2}\theta_3^{s-k}\theta_4^{k-1}g^+h^-fdxdy$$

$$- \frac{n!}{(n-s-1)!(s-k-1)!(k-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1^{n-s-1}\theta_3^{s-k-1}\theta_4^{k-1}g^+h^-fdxdy$$

$$+ \frac{n!}{(n-s-1)!(s-k)!(k-2)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1^{n-s-1}\theta_3^{s-k}\theta_4^{k-2}g^+h^-fdxdy$$

$$+ \frac{n!}{(n-s-1)!(s-k)!(k-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1^{n-s-1}\theta_3^{s-k}\theta_4^{k-1}\frac{d}{d\rho}(g^+h^-)dxdy\,. \tag{3.10}$$

From the forms in Lemma 3.1, we can see the following:

$$\left.\begin{aligned} \frac{d}{d\rho}g^+ &= \frac{y-\rho x}{1-\rho^2}f(x,y) = \frac{y-\rho x}{(1-\rho^2)^{3/2}}\phi(y)\phi\left(\frac{\rho y-x}{\sqrt{1-\rho^2}}\right) \\ &\text{and} \\ \frac{d}{d\rho}h^- &= \frac{\rho y-x}{1-\rho^2}f(x,y) = \frac{\rho y-x}{(1-\rho^2)^{3/2}}\phi(x)\phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)\,. \end{aligned}\right\} \tag{3.11}$$

Hence,

$$\frac{d}{d\rho}(g^+h^-) = \phi(x)\phi(y)\left\{ \phi(\frac{\rho y-x}{\sqrt{1-\rho^2}})\ \Phi(\frac{y-\rho x}{\sqrt{1-\rho^2}})\ \frac{y-\rho x}{(1-\rho^2)^{3/2}} \right.$$

$$\left. + \phi(\frac{y-\rho x}{\sqrt{1-\rho^2}})\ \Phi(\frac{\rho y-x}{\sqrt{1-\rho^2}})\ \frac{\rho y-x}{(1-\rho^2)^{3/2}} \right\} \qquad (3.12)$$

Equation (3.10) has singularities at the points $\rho = \pm 1$. In the next section, we will derive an approximate form of the derivative when ρ is close to 1 .

C. Approximate Formula for ${}_n\Pi'_{s:k}(\rho)$ When ρ Is Near 1

In the previous section, we derived the exact derivative of ${}_n\Pi_{s:k}(\rho)$ for $-1 < \rho < 1$. Since ${}_n\Pi_{s:k}(1) = 1 = \lim_{\rho\to 1} {}_n\Pi_{s:k}(\rho)$, the function ${}_n\Pi_{s:k}(\rho)$ is continuous on $-1 < \rho \leq 1$. Now let

$$\frac{d}{d\rho}\,{}_n\Pi_{s:k}(\rho) = I_1(\rho) + I_2(\rho) + I_3(\rho) + I_4(\rho) \qquad (3.13)$$

where $I_i(\rho)$ are the integrals of (3.10) in order. We will show that ${}_n\Pi'_{s:k}(\rho)$ is singular at $\rho = 1$ for some s and k . Therefore, in this case we cannot directly apply Taylor series approximation (first order) to ${}_n\Pi_{s:k}(\rho)$ when ρ is near 1 . Instead, we will consider a special method to resolve this difficulty. In advance we now investigate the behavior of ${}_n\Pi'_{s:k}(\rho)$ at the points near $\rho = 1$.

At first, we will consider $I_4(\rho)$ when ρ is near 1. From (3.10) and (3.12), we have

$$I_4(\rho) = \frac{n!}{(n-s-1)!(s-k)!(k-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-s-1}(x,y)\theta_3^{s-k}(x,y)\theta_4^{k-1}(x,y)\phi(x)\phi(y)$$

$$\{\phi(\frac{\rho y-x}{\sqrt{1-\rho^2}})\Phi(\frac{y-\rho x}{\sqrt{1-\rho^2}}) \frac{y-\rho x}{(\sqrt{1-\rho^2})^3} + \phi(\frac{y-\rho x}{\sqrt{1-\rho^2}})\Phi(\frac{\rho y-x}{\sqrt{1-\rho^2}}) \frac{\rho y-x}{(\sqrt{1-\rho^2})^3}\} \, dxdy \; .$$

If we take the transformation, $u = x$ and $v = (y-\rho x)/\sqrt{1-\rho^2}$, we then have

$$I_4(\rho) = \frac{n!}{(n-s-1)!(s-k)!(k-1)!} \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_1^{n-s-1}\delta_3^{s-k}\delta_4^{k-1}\phi(u)\phi(\sqrt{1-\rho^2}v+\rho u)$$

$$[\phi(\rho v-\sqrt{1-\rho^2}u)\Phi(v)v+\rho\phi(v)\Phi(\rho v-\sqrt{1-\rho^2}u)v]\,dudv$$

$$-\frac{n!}{(n-s-1)!(s-k)!(k-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_1^{n-s-1}\delta_3^{s-k}\delta_4^{k-1}\phi(u)\phi(\sqrt{1-\rho^2}v+\rho u)$$

$$\phi(v)\phi(\rho v-\sqrt{1-\rho^2}u)\,ududv$$

$$= \frac{1}{\sqrt{1-\rho^2}} k_1(\rho) - k_2(\rho) \; , \tag{3.14}$$

where $\delta_i = \theta_i(u, \sqrt{1-\rho^2}v+\rho u)$. From equation (3.14) we see that, when ρ is very close to 1 , $I_4(\rho)$ has essentially the same behavior as

$$\frac{1}{\sqrt{2}\,\sqrt{1-\rho}} \lim_{\rho\to 1} k_1(\rho) \; , \tag{3.15}$$

since $k_2(\rho)$ is bounded and dominated by $k_1(\rho)$. Before proceeding further we give the following lemmas.

Lemma 3.5: Suppose θ_i are defined by (1.3) with the standard bivariate normal p.d.f. Then, we have

(i) $\lim_{\rho\to 1} \theta_1(u,\sqrt{1-\rho^2}\, v+\rho u) = \Phi(u)$

(ii) $\lim_{\rho\to 1} \theta_2(u,\sqrt{1-\rho^2}\, v+\rho u) = \lim_{\rho\to 1} \theta_3(u,\sqrt{1-\rho^2}\, v+\rho u) = 0$

(iii) $\lim_{\rho\to 1} \theta_4(u,\sqrt{1-\rho^2}\, v+\rho u) = 1 - \Phi(u)$.

Proof: From Lemma 3.2, we have

$$\theta_1(u,\sqrt{1-\rho^2}\, v+\rho u) = \int_{-\infty}^{u} \phi(x)\Phi[v + \frac{\rho}{\sqrt{1-\rho^2}} (u-x)]\, dx .$$

Hence,

$$\lim_{\rho\to 1} \theta_1(u,\sqrt{1-\rho^2}\, v+\rho u) = \int_{-\infty}^{u} \phi(x) \lim_{\rho\to 1} \Phi[v + \frac{\rho}{\sqrt{1-\rho^2}} (u-x)]\, dx$$

$$= \int_{-\infty}^{u} \phi(x)\, dx \quad (\text{since } u-x > 0)$$

$$= \Phi(u) .$$

The others can be proved similarly. Q.E.D.

Lemma 3.6: With the same assumptions in Lemma 3.5, we have

(i) $\lim_{\rho\to 1} \frac{1}{\sqrt{1-\rho^2}} \theta_3(u,\sqrt{1-\rho^2}\, v+\rho u) = \phi(u)[\phi(v) + v\Phi(v)]$

(ii) $\lim_{\rho\to 1} \frac{1}{\sqrt{1-\rho^2}} \theta_2(u,\sqrt{1-\rho^2}\, v+\rho u) = \phi(u)[\phi(v) - v\Phi(-v)]$.

Proof: From Lemma 3.2 (iii), we have

$$\frac{1}{\sqrt{1-\rho^2}}\theta_3(u,\sqrt{1-\rho^2}\,v+\rho u) = \int_{-\infty}^{\sqrt{1-\rho^2}\,v+\rho u}\frac{1}{\sqrt{1-\rho^2}}\phi(t)\Phi(\frac{\rho t-u}{\sqrt{1-\rho^2}})\,dt$$

$$= \frac{1}{\rho}\int_{-\infty}^{\rho v-\sqrt{1-\rho^2}\,u}\phi(\frac{\sqrt{1-\rho^2}\,z+u}{\rho})\Phi(z)dz .$$

Hence, we have (i), since

$$\int_{-\infty}^{v}\Phi(z)dz = \phi(v) + v\Phi(v) .$$

Similarly, we have (ii), from Lemma 3.2 (ii). Q.E.D.

From Lemma 3.5, we know that $\lim_{\rho\to 1} k_1(\rho) \neq 0$ if $k = s$ and $\lim_{\rho\to 1} k_1(\rho) = 0$ if $k < s$ because $\lim_{\rho\to 1}\theta_3(u,\sqrt{1-\rho^2}\,v+\rho u) = 0$ (Lemma 3.5 (iii)). Hence, if $k = s$, then we have for $\rho \cong 1$,

$$I_4(\rho) = \frac{1}{\sqrt{2}\,\sqrt{1-\rho}}\lim_{\rho\to 1} k_1(\rho)$$

$$= \frac{1}{\sqrt{2}\,\sqrt{1-\rho}}\frac{n!}{(n-s-1)!(s-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(u)^{n-s-1}[1-\Phi(u)]^{s-1}$$

$$\phi^2(u)\,2\Phi(v)\phi(v)\,v\,du\,dv$$

$$= \frac{n}{\sqrt{2\pi}\,\sqrt{1-\rho}}E[\phi(Z_{n-s:n-1})] , \tag{3.16}$$

since

$$\int_{-\infty}^{\infty}\Phi(v)\phi(v)v\,dv = \frac{1}{2\sqrt{\pi}} .$$

If $k = s-1$, then from Lemma 3.6 and the first equality in (3.14), we have

$$I_4(\rho) \cong \frac{n!}{(n-s-1)!(s-2)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(u)^{n-s-1} [1-\Phi(u)]^{s-2}$$

$$\phi^3(u)[\phi(v)+v\Phi(v)]2\Phi(v)\phi(v)v\,du\,dv$$

$$= \frac{2\cdot n!}{(n-s-1)!(s-2)!} \left(\frac{\sqrt{3}}{4\pi} + \frac{1}{3}\right) \int_{-\infty}^{\infty} \Phi(u)^{n-s-1} [1-\Phi(u)]^{s-2} \phi^3(u)\,du$$

$$= n(n-1) \left(\frac{\sqrt{3}}{2\pi} + \frac{2}{3}\right) E[\phi^2(Z_{n-s:n-2})] , \tag{3.17}$$

since

$$\int_{-\infty}^{\infty} [\Phi(v)\phi^2(v)v + \Phi^2(v)\phi(v)v^2]\,dv = \frac{\sqrt{3}}{4\pi} + \frac{1}{3} .$$

Note that integrals of the above type have been collected in Owen (1980).

When $k < s-1$, since $\theta_3^{s-k}(u, \sqrt{1-\rho^2}\, v+\rho u)/\sqrt{1-\rho^2}$ goes to zero as ρ goes to 1 , we can neglect this value. Hence, we can approximate $I_4(\rho)$, when ρ is near 1 , as the following:

$$I_4(\rho) = \begin{cases} \dfrac{n}{\sqrt{2\pi}\,\sqrt{1-\rho}} E[\phi(Z_{n-s:n-1})] & \text{if } 1 \le k = s \le n-1 \\ n(n-1)\left(\dfrac{\sqrt{3}}{2\pi} + \dfrac{2}{3}\right) E[\phi^2(Z_{n-s:n-2})] & \text{if } 1 \le k = s-1 \le n-2 . \end{cases} \tag{3.18}$$

Next, we will approximate $I_1(\rho)$ when $\rho \cong 1$. From (3.10) and Lemma 3.1, we have

$$I_1(\rho) = \frac{n!}{(n-s-2)!(s-k)!(k-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_1^{n-s-2}(x,y)\theta_3^{s-k}(x,y)\theta_4^{k-1}(x,y)$$

$$\phi(y)\Phi\left(\frac{\rho y-x}{\sqrt{1-\rho^2}}\right)\phi(x)\Phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) \frac{1}{\sqrt{1-\rho^2}} \phi(x)\phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) dx\,dy .$$

If we take the transformation $u = x$ and $v = (y-\rho x)/\sqrt{1-\rho^2}$, then

$$I_1(\rho) = \frac{n!}{(n-s-2)!(s-k)!(k-1)!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_1^{n-s-2}\delta_3^{s-k}\delta_4^{k-1}\phi(\sqrt{1-\rho^2}\ v+\rho u)$$

$$\Phi(\rho v-\sqrt{1-\rho^2}\ u)\ \phi^2(u)\Phi(v)\phi(v)\,du\,dv\ . \quad (3.19)$$

So, using Lemma 3.5 ,

$$I_1(\rho) \cong \begin{cases} \frac{n(n-1)}{3} E[\phi^2(Z_{n-s-1:n-2})] & \text{if } 1 \le k = s \le n-2 \\ 0 & \text{otherwise .} \end{cases} \quad (3.20)$$

With the same arguments above, we can see that

$$I_2(\rho) \cong \begin{cases} -\frac{n(n-1)}{3} E[\phi^2(Z_{n-s:n-2})] & \text{if } 1 \le k = s-1 \le n-2 \\ 0 & \text{otherwise} \end{cases} \quad (3.21)$$

and

$$I_3(\rho) \cong \begin{cases} \frac{n(n-1)}{3} E[\phi^2(Z_{n-s:n-2})] & \text{if } 2 \le k = s \le n-1 \\ 0 & \text{otherwise .} \end{cases} \quad (3.22)$$

Because of $I_4(\rho)$, ${}_n\Pi'_{s:k}(\rho)$ has a singularity at $\rho = 1$ when $1 \le k = s \le n-1$. If we combine (3.18), (3.20), and (3.22), then from (3.13), we have that for $k = s$,

$$ {}_n\Pi'_{s:s}(\rho) \cong \frac{1}{\sqrt{1-\rho}} A + B \quad (3.23)$$

where

$$A = \frac{n}{\sqrt{2\pi}} E[\phi(Z_{n-s:n-1})]$$

and

$$B = \begin{cases} \frac{n(n-1)}{3} E[\phi^2(Z_{n-2:n-2})] & \text{if } s = 1 \text{ or } s = n-1 \\ \frac{n(n-1)}{3} E[\phi^2(Z_{n-s-1:n-2}) + \phi^2(Z_{n-s:n-2})] & \\ & \text{if } 2 \leq s \leq n-2 . \end{cases}$$

Now we have the problem: what is ${}_n\Pi_{s:s}(\rho)$ satisfying (3.23) and the condition ${}_n\Pi_{s:s}(1) = 1$? The answer is easily found, i.e.,

$$\begin{aligned} {}_n\Pi_{s:s}(\rho) &= A \int^{\rho} \frac{1}{\sqrt{1-t}}\, dt + B\rho \\ &= -2A\sqrt{1-\rho} + B\rho + C \end{aligned}$$

and

$${}_n\Pi_{s:s}(1) = B + C = 1 .$$

Therefore,

$${}_n\Pi_{s:s}(\rho) \cong 1 - 2A\sqrt{1-\rho} - (1-\rho)B . \tag{3.24}$$

When $k = s-1$, $\lim_{\rho\to 1} {}_n\Pi'_{s:k}(\rho)$ is finite. Hence, from (3.18) and (3.21) we have

$${}_n\Pi_{s:s-1}(\rho) \cong 1 - (1-\rho)D , \tag{3.25}$$

where

$$D = \lim_{\rho\to 1} {}_n\Pi'_{s:s-1}(\rho) = n(n-1)\left(\frac{\sqrt{3}}{2\pi} + \frac{1}{3}\right)E[\phi^2(Z_{n-s:n-2})] .$$

From (3.24) and (3.25), we give the approximate formula of ${}_n\Pi_{s:k}(\rho)$ in one equation as the following: for ρ very close to 1 ,

$$
{}_{n}\Pi_{s:k}(\rho) = \begin{cases} 1 - \dfrac{n\sqrt{2(1-\rho)}}{\sqrt{\pi}} E[\phi(Z_{n-1:n-1})] \\ \quad - \dfrac{n(n-1)(1-\rho)}{3} E[\phi^2(Z_{n-2:n-2})] \\ \quad \text{if} \;\; s = k = 1 \;\; \text{or} \;\; s = k = n-1 \\ 1 - \dfrac{n\sqrt{2(1-\rho)}}{\sqrt{\pi}} E[\phi(Z_{n-s:n-1})] \\ \quad - \dfrac{n(n-1)(1-\rho)}{3} E[\phi^2(Z_{n-s-1:n-2}) + \phi^2(Z_{n-s:n-2})] \\ \quad \text{if} \;\; 2 \le k = s \le n-2 \\ 1 - n(n-1)(1-\rho)(\frac{\sqrt{3}}{2\pi} + \frac{1}{3}) E[\phi^2(Z_{n-s:n-2})] \\ \quad \text{if} \;\; 1 \le k = s-1 \le n-2 . \end{cases} \tag{3.26}
$$

When $k < s-1$, we need the higher-order derivatives of ${}_{n}\Pi_{s:k}(\rho)$ to obtain an approximate formula. For small $n(\le 5)$, ${}_{n}\Pi_{s:k}(\rho)$ with $k \le s-2$ are near 0.999. Hence, we do not give the approximate formula for $k \le s-2$ because of difficulties in evaluating the higher-order derivatives.

The approximate formula (3.26) underestimates the probability since we used the limiting value of the derivatives, i.e., the slope of ${}_{n}\Pi_{s:k}(\rho)$ is steeper than the actual slope. Equation (3.26) can be used to obtain the value of ρ for given P such that ${}_{n}\Pi_{s:k}(\rho) \ge P$. When we can expect that ρ is near 1, we have the equation of unknown ρ by equating the left side of (3.26) to the given value of P.

Example 3.1: Suppose $n = 5$ and $s = k = 1$. Since $E[\phi(Z_{4:4})] = 0.2326$ and $E[\phi^2(Z_{3:3})] = 0.0800$, we have the following equation for given value $P = 0.9$.

$$0.9 = 1 - \frac{5\sqrt{2}}{\sqrt{\pi}} \sqrt{1-\rho} \times 0.2326 - (1-\rho)\, \frac{20}{3} \times 0.0800 \ .$$

Putting $Z = \sqrt{1-\rho}$, we have

$$0.5333\, Z^2 + 0.9279\, Z - 0.1 = 0 \ .$$

Hence,

$$Z = \frac{-0.9279 + \sqrt{0.9279^2 + 4\times 0.1\times 0.5333}}{2\times 0.5333} = 0.1018$$

and

$$\rho = 1 - Z^2 = 0.9896 \ .$$

In Table 2.2, we see that the value of ρ corresponding to $P = 0.9$ is 0.988.

Example 3.2: Take $n = 6$ and $s = k = 2$. From the second equation in (3.26), we have

$$0.9 = 1 - \frac{6\sqrt{2}}{\sqrt{\pi}} \sqrt{1-\rho} \times 0.211201 - (1-\rho)\, \frac{6\times 5}{3}\, (0.23259 + 0.33160).$$

Hence,

$$\rho = 0.99498 \cong 0.995 \ .$$

Since ${}_6\Pi_{2:2}(0.995) = 0.8953$ from Table 2.3, we can see that $\rho = 0.995$ gives ${}_6\Pi_{2:2} \cong 0.9$ and hence, that the formula (3.26) is very useful when $k \geq 2$. In order to save the time in computation of $E[\phi(Z_{i:n})]$

and $E[\phi^2(Z_{i:n})]$, we give the values in Table 3.1 and 3.2. Note that these two expectations are both symmetric with respect to i , i.e.,

$$E[\phi(Z_{i:n})] = E[\phi(Z_{n-i+1:n})]$$

and

$$E[\phi^2(Z_{i:n})] = E[\phi^2(Z_{n-i+1:n})] \ .$$

Table 3.1 $E[\phi(Z_{i:n})]$ where $Z_{i:n}$ is i-th order statistic from the standard normal sample of size n

$$E[\phi(Z_{i:n})] = E[\phi(Z_{n-i+1:n})]$$

n\i	1	2	3	4	5	6	7	8	9	10
1	0.0									
2	0.282095									
3	0.257344	0.331597								
4	0.232593	0.331597								
5	0.211201	0.318160	0.351751							
6	0.193168	0.301365	0.351751							
7	0.177950	0.284478	0.343581	0.362645						
8	0.165001	0.268590	0.332142	0.362645						
9	0.153875	0.254011	0.319617	0.357193	0.369460					
10	0.144221	0.240759	0.307017	0.349016	0.369460					
11	0.135769	0.228747	0.294817	0.339553	0.365574	0.374123				
12	0.128307	0.217851	0.283223	0.329596	0.359468	0.374123				
13	0.121670	0.207949	0.272315	0.319584	0.352124	0.371217	0.377514			
14	0.115728	0.198923	0.262103	0.309761	0.344141	0.366494	0.377514			
15	0.110374	0.190671	0.252563	0.300261	0.335886	0.360650	0.375260	0.380090		
16	0.105526	0.183101	0.243658	0.291151	0.327590	0.354139	0.371503	0.380090		
17	0.101113	0.176136	0.235343	0.282461	0.319394	0.347260	0.366751	0.378292	0.382114	
18	0.097078	0.169706	0.227572	0.274197	0.311387	0.340213	0.361352	0.375234	0.382114	
19	0.093374	0.163754	0.220301	0.266350	0.303620	0.333134	0.355551	0.371298	0.380646	0.383745
20	0.089960	0.158228	0.213488	0.258908	0.296122	0.326113	0.349517	0.366757	0.378109	0.383745
21	0.086804	0.153084	0.207095	0.251849	0.288907	0.319210	0.343372	0.361808	0.374798	0.382524
	0.385089°									
22	0.083877	0.148284	0.201086	0.245154	0.281978	0.312463	0.337201	0.356596	0.370929	0.380387
	0.385089									
23	0.081153	0.143794	0.195429	0.238800	0.275334	0.305898	0.331064	0.351227	0.366663	0.377564
	0.384057	0.386214								
24	0.078613	0.139585	0.190095	0.232766	0.268967	0.299529	0.325005	0.345779	0.362123	0.374230
	0.382233	0.386214								
25	0.076237	0.135630	0.185058	0.227033	0.262868	0.293363	0.319053	0.340310	0.357401	0.370518
	0.379798	0.385331	0.387170							
26	0.074010	0.131908	0.180294	0.221580	0.257026	0.287404	0.313229	0.334863	0.352567	0.366531
	0.376896	0.383755	0.387170							
27	0.071918	0.128399	0.175782	0.216388	0.251430	0.281649	0.307545	0.329468	0.347675	0.362351
	0.373638	0.381635	0.386406	0.387992						
28	0.069949	0.125083	0.171503	0.211442	0.246067	0.276096	0.302010	0.324150	0.342765	0.358039
	0.370112	0.379087	0.385032	0.387992						
29	0.068092	0.121945	0.167439	0.206724	0.240927	0.270740	0.296629	0.318923	0.337869	0.353645
	0.366389	0.376205	0.383169	0.387325	0.388706					
30	0.066338	0.118972	0.163574	0.202221	0.235998	0.265575	0.291402	0.313801	0.333010	0.349206
	0.362523	0.373066	0.380915	0.386117	0.388706					

°continued for i=11, 12 , ...

Table 3.1 (continued)

n\i	1	2	3	4	5	6	7	8	9	10
31	0.064677 0.358559	0.116150 0.369729	0.159895 0.378348	0.197917 0.384469	0.231268 0.388119	0.260593 0.389331	0.286331	0.308791	0.328206	0.344753
32	0.063103 0.354533	0.113467 0.366246	0.156387 0.375534	0.193802 0.382460	0.226727 0.387051	0.255789 0.389331	0.281412	0.303897	0.323471	0.340307
33	0.061609 0.350471	0.110914 0.362656	0.153040 0.372528	0.189862 0.380160	0.222365 0.385583	0.251154 0.388812	0.276643 0.389882	0.299124	0.318814	0.335888
34	0.060189 0.346396	0.108481 0.358992	0.149842 0.369374	0.186087 0.377623	0.218172 0.383783	0.246683 0.387862	0.272021 0.389882	0.294471	0.314244	0.331509
35	0.058837 0.342325	0.106160 0.355279	0.146783 0.366110	0.182468 0.374899	0.214139 0.381709	0.242367 0.386549	0.267542 0.389421	0.289940 0.390369	0.309765	0.327183
36	0.057548 0.338272	0.103942 0.351536	0.143855 0.362764	0.178994 0.372029	0.210258 0.379409	0.238201 0.384929	0.263200 0.388573	0.285528 0.390369	0.305381	0.322917
37	0.056318 0.334250	0.101822 0.347780	0.141048 0.359360	0.175658 0.369047	0.206521 0.376927	0.234177 0.383049	0.258992 0.387395	0.281234 0.389960	0.301094 0.390801	0.318720
38	0.055143 0.330267	0.099792 0.344024	0.138357 0.355917	0.172450 0.365982	0.202920 0.374302	0.230289 0.380953	0.254913 0.385932	0.277056 0.389203	0.296903 0.390801	0.314596
39	0.054019 0.326333	0.097847 0.340280	0.135773 0.352448	0.169365 0.362855	0.199448 0.371566	0.226530 0.378679	0.250959 0.384222	0.272990 0.388144	0.292810 0.390438	0.310549 0.391184
40	0.052943 0.322455	0.095982 0.336558	0.133290 0.348965	0.166394 0.359683	0.196098 0.368747	0.222896 0.376264	0.247125 0.382302	0.269035 0.386820	0.288812 0.389763	0.306581 0.391184
41	0.051912 0.318637 0.391521	0.094191 0.332868	0.130903 0.345478	0.163533 0.356476	0.192864 0.365866	0.219380 0.373740	0.243406 0.380207	0.265187 0.385260	0.284908 0.388813	0.302691 0.390862
42	0.050923 0.314883 0.391521	0.092470 0.329216	0.128605 0.341996	0.160774 0.353245	0.189741 0.362939	0.215977 0.371136	0.239799 0.377972	0.261443 0.383494	0.281097 0.387615	0.298882 0.390264
43	0.049973 0.311195 0.391239	0.090816 0.325611 0.391816	0.126392 0.338529	0.158113 0.349997	0.186722 0.359973	0.212682 0.368473	0.236298 0.375630	0.257801 0.381553	0.277377 0.386189	0.295150 0.389415
44	0.049060 0.307574 0.390713	0.089223 0.322058 0.391816	0.124259 0.335086	0.155544 0.346739	0.183804 0.356977	0.209489 0.365766	0.232901 0.373211	0.254258 0.379472	0.273746 0.384560	0.291496 0.388334
45	0.048183 0.304019 0.389961	0.087689 0.318561 0.391573	0.122201 0.331676 0.392070	0.153062 0.343480	0.180980 0.353955	0.206395 0.363023	0.229602 0.370739	0.250810 0.377284	0.270203 0.382753	0.287919 0.387032
46	0.047337 0.300530 0.388990	0.086211 0.315122 0.391117	0.120215 0.328306 0.392070	0.150664 0.340230	0.178246 0.350910	0.203394 0.360247	0.226397 0.368228	0.247455 0.375022	0.266746 0.380803	0.284416 0.385525
47	0.046523 0.297104 0.387804	0.084785 0.311740 0.390458	0.118297 0.324983 0.391865	0.148344 0.336997 0.392285	0.175599 0.347849	0.200484 0.357439	0.223283 0.365686	0.244190 0.372712	0.263373 0.378744	0.280986 0.383837

Table 3.1 (continued)

n\i	1	2	3	4	5	6	7	8	9	10
48	0.045739	0.083408	0.116444	0.146100	0.173034	0.197659	0.220256	0.241012	0.260083	0.277630
	0.293741	0.308417	0.321711	0.333793	0.344780	0.354601	0.363116	0.370373	0.376612	0.381998
	0.386412	0.389595	0.391478	0.392285						
49	0.044981	0.082079	0.114652	0.143927	0.170547	0.194917	0.217313	0.237917	0.256873	0.274346
	0.290438	0.305149	0.318493	0.330625	0.341713	0.351734	0.360515	0.368012	0.374438	0.380045
	0.384829	0.388522	0.390913	0.392117	0.392460					
50	0.044251	0.080794	0.112917	0.141821	0.168135	0.192253	0.214450	0.234902	0.253743	0.271133
	0.287196	0.301934	0.315328	0.327501	0.338658	0.348842	0.357879	0.365632	0.372243	0.378018
	0.383086	0.387236	0.390157	0.391800	0.392460					

Table 3.2 $E(\phi^2(Z_{i:n}))$ where $Z_{i:n}$ is i-th order statistic from the standard normal sample of size n

$$E(\phi^2(Z_{i:n})) = E(\phi^2(Z_{n-i+1:n}))$$

n\i	1	2	3	4	5	6	7	8	9	10
1	0.0									
2	0.091888									
3	0.080002	0.115660								
4	0.068116	0.115660								
5	0.058110	0.108141	0.126940							
6	0.049983	0.098741	0.126940							
7	0.043406	0.089448	0.121976	0.133558						
8	0.038047	0.080921	0.115027	0.133558						
9	0.033637	0.073326	0.107504	0.130072	0.137916					
10	0.029970	0.066639	0.100074	0.124842	0.137916					
11	0.026890	0.060773	0.093036	0.118842	0.135342	0.141004				
12	0.024277	0.055626	0.086509	0.112615	0.131298	0.141004				
13	0.022042	0.051098	0.080525	0.106458	0.126466	0.139030	0.143307			
14	0.020114	0.047104	0.075068	0.100534	0.121269	0.135821	0.143307			
15	0.018439	0.043565	0.070104	0.094922	0.115968	0.131871	0.141746	0.145091		
16	0.016974	0.040418	0.065592	0.089655	0.110722	0.127508	0.139144	0.145091		
17	0.015684	0.037609	0.061489	0.084740	0.105628	0.122949	0.135866	0.143827	0.146514	
18	0.014543	0.035091	0.057754	0.080168	0.100739	0.118339	0.132169	0.141676	0.146514	
19	0.013527	0.032825	0.054347	0.075924	0.096084	0.113773	0.128232	0.138918	0.145469	0.147676
20	0.012619	0.030780	0.051234	0.071987	0.091674	0.109313	0.124181	0.135754	0.143664	0.147676
21	0.011804	0.028927	0.048384	0.068334	0.087511	0.104996	0.120105	0.132333	0.141313	0.146798
	0.148641									
22	0.011069	0.027243	0.045769	0.064944	0.083590	0.100845	0.116065	0.128763	0.138580	0.145261
	0.148641									
23	0.010403	0.025707	0.043365	0.061795	0.079900	0.096872	0.112102	0.125124	0.135587	0.143236
	0.147894	0.149457								
24	0.009799	0.024304	0.041150	0.058868	0.076431	0.093082	0.108243	0.121472	0.132427	0.140854
	0.146571	0.149457								
25	0.009248	0.023016	0.039105	0.056144	0.073171	0.089474	0.104508	0.117849	0.129170	0.138218
	0.144809	0.148813	0.150155							
26	0.008745	0.021833	0.037214	0.053605	0.070105	0.086044	0.100905	0.114285	0.125868	0.135407
	0.142716	0.147662	0.150155							
27	0.008283	0.020743	0.035461	0.051237	0.067223	0.082788	0.097442	0.110801	0.122561	0.132482
	0.140379	0.146116	0.149594	0.150760						
28	0.007859	0.019736	0.033834	0.049024	0.064511	0.079697	0.094119	0.107410	0.119278	0.129491
	0.137865	0.144264	0.148585	0.150760						
29	0.007468	0.018804	0.032320	0.046955	0.061958	0.076765	0.090936	0.104122	0.116042	0.126471
	0.135229	0.142180	0.147218	0.150267	0.151287					
30	0.007107	0.017939	0.030909	0.045017	0.059553	0.073984	0.087891	0.100941	0.112867	0.123450
	0.132513	0.139920	0.145568	0.149375	0.151287					

Table 3.2 (continued)

n\i	1	2	3	4	5	6	7	8	9	10
31	0.006773	0.017136	0.029592	0.043199	0.057286	0.071345	0.084980	0.097872	0.109765	0.120449
	0.129751	0.137534	0.143698	0.148158	0.150852	0.151752				
32	0.006463	0.016387	0.028361	0.041492	0.055146	0.068840	0.082198	0.094915	0.106744	0.117486
	0.126969	0.135060	0.141658	0.146680	0.150058	0.151752				
33	0.006174	0.015689	0.027209	0.039888	0.053125	0.066462	0.079541	0.092069	0.103809	0.114571
	0.124189	0.132529	0.139490	0.144994	0.148969	0.151364	0.152163			
34	0.005906	0.015037	0.026128	0.038378	0.051215	0.064204	0.077003	0.089332	0.100964	0.111714
	0.121427	0.129966	0.137229	0.143142	0.147638	0.150655	0.152163			
35	0.005655	0.014426	0.025113	0.036955	0.049408	0.062058	0.074578	0.086702	0.098209	0.108922
	0.118694	0.127390	0.134903	0.141164	0.146110	0.149676	0.151817	0.152529		
36	0.005421	0.013853	0.024158	0.035612	0.047696	0.060017	0.072261	0.084175	0.095545	0.106200
	0.116000	0.124816	0.132537	0.139090	0.144423	0.148472	0.151181	0.152529		
37	0.005202	0.013316	0.023259	0.034344	0.046075	0.058075	0.070048	0.081748	0.092972	0.103551
	0.113353	0.122257	0.130148	0.136948	0.142610	0.147082	0.150298	0.152219	0.152855	
38	0.004996	0.012811	0.022412	0.033144	0.044536	0.056227	0.067932	0.079418	0.090487	0.100977
	0.110759	0.119721	0.127750	0.134759	0.140701	0.145537	0.149205	0.151647	0.152855	
39	0.004803	0.012335	0.021612	0.032009	0.043076	0.054467	0.065909	0.077180	0.088090	0.098478
	0.108222	0.117217	0.125355	0.132540	0.138720	0.143869	0.147935	0.150849	0.152578	0.153147
40	0.004621	0.011886	0.020857	0.030934	0.041688	0.052789	0.063974	0.075030	0.085777	0.096056
	0.105745	0.114750	0.122972	0.130306	0.136690	0.142104	0.146516	0.149855	0.152063	0.153147
41	0.004450	0.011463	0.020142	0.029914	0.040369	0.051189	0.062123	0.072966	0.083546	0.093709
	0.103332	0.112327	0.120607	0.128066	0.134626	0.140267	0.144975	0.148692	0.151341	0.152899
	0.153408									
42	0.004289	0.011063	0.019464	0.028946	0.039113	0.049662	0.060351	0.070983	0.081395	0.091436
	0.100983	0.109952	0.118266	0.125827	0.132542	0.138378	0.143337	0.147384	0.150437	0.152436
	0.153408									
43	0.004137	0.010684	0.018822	0.028026	0.037917	0.048204	0.058654	0.069077	0.079319	0.089235
	0.098699	0.107627	0.115956	0.123597	0.130447	0.136454	0.141625	0.145955	0.149370	0.151784
	0.153186	0.153640								
44	0.003993	0.010326	0.018213	0.027150	0.036776	0.046811	0.057028	0.067246	0.077317	0.087104
	0.096480	0.105357	0.113682	0.121380	0.128347	0.134507	0.139860	0.144427	0.148161	0.150961
	0.152772	0.153640								
45	0.003857	0.009986	0.017634	0.026318	0.035689	0.045479	0.055470	0.065486	0.075386	0.085041
	0.094324	0.103142	0.111447	0.119182	0.126247	0.132546	0.138061	0.142825	0.146830	0.149982
	0.152185	0.153443	0.153846							
46	0.003728	0.009663	0.017083	0.025524	0.034650	0.044204	0.053976	0.063794	0.073524	0.083044
	0.092231	0.100985	0.109256	0.117008	0.124152	0.130577	0.136238	0.141170	0.145400	0.148862
	0.151437	0.153075	0.153846							
47	0.003605	0.009357	0.016560	0.024767	0.033658	0.042984	0.052543	0.062167	0.071727	0.081110
	0.090198	0.098883	0.107113	0.114863	0.122065	0.128604	0.134401	0.139479	0.143894	0.147620
	0.150539	0.152548	0.153673	0.154026						

Table 3.2 (continued)

n\i	1	2	3	4	5	6	7	8	9	10
48	0.003489	0.009065	0.016061	0.024045	0.032709	0.041815	0.051167	0.060602	0.069993	0.079238
	0.088224	0.096838	0.105019	0.112751	0.119991	0.126628	0.132556	0.137767	0.142333	0.146276
	0.149502	0.151872	0.153348	0.154026						
49	0.003379	0.008788	0.015585	0.023356	0.031802	0.040694	0.049846	0.059096	0.068321	0.077426
	0.086307	0.094848	0.102975	0.110677	0.117935	0.124652	0.130703	0.136043	0.140737	0.144852
	0.148340	0.151051	0.152880	0.153876	0.154183					
50	0.003274	0.008524	0.015131	0.022697	0.030933	0.039620	0.048576	0.057646	0.066707	0.075671
	0.084444	0.092911	0.100983	0.108646	0.115901	0.122679	0.128844	0.134311	0.139121	0.143372
	0.147073	0.150091	0.152273	0.153593	0.154183					

IV. APPLICATIONS AND EXAMPLES

A. Components of Variance Model

The large literature on selection procedures is confined to fixed effects models (Model I). In the indifference zone approach, the problem is to determine the minimum sample size required to ensure that the probability of correct selection of the best population is at least equal to a given value P^* for the least favorable configuration. In the subset selection approach, the size of the selected subset is determined so as to include the best population. Hence, the determination of the sample size and/or the subset size are the most important factors in selection problems.

By the theory in Chapter II, interesting and useful connections can be established with the components of variance model (Model II); see e.g., Scheffé (1959). As in the fixed effects model, the goal is to select the s (random) "treatments" ensuring that the probability of including the k ($\leq$ s) best treatment is at least equal to a given value P^*. First, let us consider a balanced one-way components of variance model:

$$X_{ij} = \mu + T_i + Z_{ij}, \quad (i = 1, 2, \ldots, n \ ; \ j = 1, 2, \ldots, r) \tag{4.1}$$

where, as usual, the T_i are normal $N(0, \sigma_T^2)$, the Z_{ij} are $N(0, \sigma_Z^2)$, and all T's and Z's are mutually independent. Define the probability

$$\begin{aligned} P = \Pr\{\{\text{rank } T_{i_1}, \ldots, \text{rank } T_{i_s}\} \supset \{n, n-i, \ldots, n-k+i\} \mid \\ \{\text{rank } \bar{X}_{i_1}, \ldots, \text{rank } \bar{X}_{i_s}\} = \{n, n-1, \ldots, n-s+1\}\} \end{aligned} \tag{4.2}$$

which is the probability that, given the s largest $\bar{X}_i$, the corresponding s T_i include the k largest T_i. Suppose we know the variances σ_T^2 and σ_Z^2, or the variance ratio $\zeta = \sigma_T^2/\sigma_Z^2$. Then the probability P is a function of ρ, the correlation between $\bar{X}_{i.}$ and T_i, as well as of k, s, and n. But the correlation can be expressed by

$$\begin{aligned}\rho &= \mathrm{Corr.}(\bar{X}_{i.}, T_i) \\ &= \frac{\sqrt{\sigma_T^2}}{\sqrt{\sigma_T^2 + \sigma_Z^2/r}} \\ &= [1 + \frac{1}{r\zeta}]^{-\frac{1}{2}} \,. \end{aligned} \tag{4.3}$$

When n, the number of treatments, is fixed, we can make P acceptably large for given values of ζ, s, and k by increasing r, the number of replications. If n, k, r, and ζ are given, we can determine the subset size s in order that ${}_n\Pi_{s:k}(\rho) \geq P^*$.

From (4.3), we see that for given P^*, and hence, fixed ρ, the (unrounded) value of r is inversely proportional to ζ. After finding r such that the probability of correct selection is at least P^*, we can compute, from the point of view of analysis of variance, the power of testing

$$H_0: \zeta = 0 \quad \text{vs.} \quad H_1: \zeta = \zeta_1 > 0 \,. \tag{4.4}$$

We know that

$$R = \frac{r \sum_{i=1}^{n} (\bar{X}_{i.} - \bar{X}_{..})^2/(n-1)}{\sum_{i=1}^{n} \sum_{j=1}^{r} (X_{ij} - \bar{X}_{i.})^2/n(r-1)} \tag{4.5}$$

is distributed as the ratio

$$\frac{[(\sigma_Z^2 + r\sigma_T^2)\ \chi_{n-1}^2]/(n-1)}{\sigma_Z^2\ \chi_{n(r-1)}^2/n(r-1)}$$

where the two χ^2's are mutually independent. Hence, we may write

$$R = (1 + r\zeta)\ F_{n-1,n(r-1)}\ , \tag{4.6}$$

where $F_{\ell,m}$ is a central F-variate with ℓ and m d.f.; that is, R is distributed as a multiple of a central F-variate. This multiple is unity under H_0 and $1 + r\zeta_1$ under H_1. One may test H_0 by means of the following decision rule:

$$\text{Reject } H_0\colon \zeta = 0 \text{ if } R \geq F_{1-\alpha;n-1,n(r-1)}\ , \tag{4.7}$$

for chosen level of significance α. The power function of the test is given by

$$\begin{aligned}\beta(\zeta) &= \Pr\{R \geq F_{1-\alpha;n-1,n(r-1)}\} \\ &= \Pr\{F_{n-1,n(r-1)} \geq F_{1-\alpha;n-1,n(r-1)}/(1 + r\zeta)\} \\ &= I_{x_0}(a,b)\ ,\end{aligned} \tag{4.8}$$

where $a = \frac{1}{2}\, n(r-1)$, $b = \frac{1}{2}\,(n-1)$, $x_0 = a/(a+bF_0)$, $F_0 = F_{1-\alpha;n-1,n(r-1)}/(1 + r\zeta)$, and $I_{x_0}(a,b) = \int_0^{x_0} x^{a-1}(1-x)^{b-1}dx/B(a,b)$, the incomplete beta-function tabulated in Pearson (1934).

We begin with an example illustrating the theory developed in Chapter II before proceeding to examples relating to the components of variance model. This is a generalization of the example in David et al. (1977).

Example 4.1: Suppose that the scores of candidates taking two tests are bivariate normal with $\rho = 0.9$. Out of 9 candidates taking the first (screening) test, the top s are selected and given the second test. What is the smallest value of s ensuring with probability at least 0.95 that the best $k(\leq s)$ of the 9 candidates, as judged by the second test, are included among the s selected? The answers can be read off from Table 2.4 when $k = 1$ and obtained from Table 2.3 for $k \geq 2$. Corresponding to $k = 1, 2, 3$, we see that $s = 4, 5, 7$.

The following two examples show how to determine the number of replications so that the probability of correct selection is at least equal to a given value and how to compute the power of the test (4.4).

Example 4.2: Let T_i be a (genotypic) effect of interest and $\bar{X}_i$ the corresponding (phenotypic) sample mean. Suppose $n = 6$ and $\zeta_1 = \sigma_T^2/\sigma_Z^2 = 1.0$. The problem is to select the largest effect with the probability of correct selection at least equal to 0.9 . In this case, we have $s = 1$ and $k = 1$. From Table 2.2, we have that the value of ρ should be greater than or equal to 0.9898. Hence, by (4.3), we have

$$[1 + \frac{1}{r}]^{-\frac{1}{2}} \geq 0.9898$$

or

$$r \geq 48.27 \ .$$

In other words, we have to choose at least 49 replications in order to select the largest effect with probability at least equal to 0.9 .

If the subset size $s = 2$, we can see from Table 2.3 that $\rho \geq 0.9$. Hence, we have

$$[1 + \frac{1}{r}]^{-\frac{1}{2}} \geq 0.9$$

or

$$r \geq 4.26$$

which implies that using subset size $s = 2$, 5 replications are enough to select the largest genotypic effect with probability at least equal to 0.9 . It may be noted that at significance level 0.05 the power $\beta(1)$ of the test H_0 vs. H_1 in (4.4) is from (4.8)

$$\beta(1) = \Pr\{F_{5,288} \geq 0.046\} \cong 0.997 ,$$

in the first case, and

$$\beta(1) = \Pr\{F_{5,24} \geq 0.437\} \cong 0.818 ,$$

in the second case. Note that the power in the first case is computed by using the following approximation formula (see Abramowitz and Stegun, 1972, p. 947).

$$\Pr\{F_{\ell,m} \geq x\} \cong \Pr\{Z \geq z\}$$

where Z is the standard normal variate and

$$z = \frac{x^{1/3}(1 - \frac{2}{9m}) - (1 - \frac{2}{9\ell})}{\sqrt{\frac{2}{9\ell} + x^{2/3}\frac{2}{9m}}} .$$

<u>Example 4.3</u>: T_i represents a machine effect, the 10 machines being drawn from a large pool of machines of the same make but with the usual

differences in performance over machines and weeks as measured by X_{ij} . We want to select the two best machines with probability at least equal to 0.8 . Suppose that $\zeta = \sigma_T^2/\sigma_Z^2 = 3$. Since $s = k = 2$, by Table 2.3 we have $\rho \geq 0.99$. Hence, from (4.3) we have

$$[1 + \frac{1}{3r}]^{-\frac{1}{2}} \geq 0.99$$

which implies that $r \geq 16.4$. That is, we have to test the machines for 17 weeks.

In Example 4.3, X_{ij} might be the number of hours machine i works satisfactorily in week j . We may actually be interested in choosing the two worst machines out of the ten in order to replace these or because of a reduction in work load.

B. Other Models

In this section, we will consider components of variance models (balanced) other than the one-way case. In the two-way classification the observation in the (i,j) cell is assumed to be

$$X_{ij} = \mu + T_i + R_j + Z_{ij} ,$$
$$(i = 1, \ldots, n; \quad j = 1, \ldots, m) \tag{4.9}$$

where the T_i are normal $N(0, \sigma_T^2)$, the R_i are $N(0, \sigma_R^2)$, the Z_{ij} are $N(0, \sigma_Z^2)$, and all T's, R's, and Z's are independent. To select the largest k of the T_i, we note that

$$\bar{X}_{i.} = \mu + T_i + \bar{R}_{.} + \bar{Z}_{i.}$$

and

$$\rho = \text{Corr.}(\bar{X}_{i.}, T_i)$$
$$= (1 + \frac{1}{m\zeta})^{-\frac{1}{2}} , \qquad (4.10)$$

where $\zeta = \sigma_T^2/(\sigma_R^2 + \sigma_Z^2)$.

If the model (4.10) has r replications, i.e.,

$$X_{ijk} = \mu + T_i + R_j + Z_{ijk} ,$$
$$(i = 1, \ldots, n ; \ j = 1, \ldots, m ; \ k = 1, \ldots, r), \qquad (4.11)$$

then we have

$$\bar{X}_{i..} = \mu + T_i + \bar{R}_{.} + \bar{Z}_{i..}$$

and

$$\rho = \text{Corr.}(\bar{X}_{i..}, T_i)$$
$$= (1 + \frac{1}{mr\zeta})^{-\frac{1}{2}} \qquad (4.12)$$

where $\zeta = \sigma_T^2/(r\sigma_R^2 + \sigma_Z^2)$. Note that the model (4.11) is the balanced two-way components of variance model without interactions. Under the hypothesis H_0: $\sigma_T^2 = 0$, SS_T is distributed as $\sigma_Z^2 \chi_{n-1}^2$. For known σ_T^2/σ_Z^2 , from the relation

$$\frac{MS_T}{MS_Z} = \frac{\sigma_Z^2 + mr\sigma_T^2}{\sigma_Z^2} F_{n-1,nmr-n-m-1}$$

we can evaluate the power for testing H_0: $\frac{\sigma_T^2}{\sigma_Z^2} = 0$ vs. H_1: $\frac{\sigma_T^2}{\sigma_Z^2} = \delta_1^2 > 0.$

Similarly, to the procedure in the previous section, we can determine the number of replications.

If the model (4.11) has interactions, then this model is expressed by

$$X_{ijk} = \mu + T_i + R_j + S_{ij} + Z_{ijk} ,$$
$$(i = 1, \ldots, n \; ; \; j = 1, \ldots, m \; ; \; k = 1, \ldots, r), \quad (4.13)$$

with the usual assumptions. Similarly, as before, we have

$$\begin{aligned} \rho &= \text{Corr.}(\bar{X}_{i..} , T_i) \\ &= (1 + \frac{1}{mr\zeta})^{-\frac{1}{2}} \end{aligned} \quad (4.14)$$

where $\zeta = \sigma_T^2/(r\sigma_R^2 + r\sigma_S^2 + \sigma_Z^2)$. Since

$$\frac{MS_T}{MS_S} = \frac{\sigma_Z^2 + r\sigma_S^2 + mr\sigma_T^2}{\sigma_Z^2 + r\sigma_S^2} F_{n-1,(n-1)(m-1)} ,$$

we have the power function for testing

$$H_0: \frac{\sigma_T^2}{\sigma_Z^2 + r\sigma_S^2} = 0 \quad \text{vs.} \quad H_1: \frac{\sigma_T^2}{\sigma_Z^2 + r\sigma_S^2} = \delta_1^2 > 0 .$$

In models (4.10) and (4.13), we can choose r , for given variance ratios, so that the probability of selection is at least P^* .

The following gives a list of other models with the corresponding formula for ρ .

(i) Two-way mixed model without interactions:

$$X_{ijk} = \alpha_i + T_j + Z_{ijk} \quad \begin{pmatrix} i = 1, \ldots, m; \\ j = 1, \ldots, n; \\ k = 1, \ldots, r \end{pmatrix}$$

$$\rho = (1 + \frac{1}{mr\zeta})^{-\frac{1}{2}} , \; \zeta = \sigma_T^2/\sigma_Z^2$$

(ii) Two-way mixed model with interactions:

$$X_{ijk} = \alpha_i + T_j + S_{ijk} + Z_{ijk} \quad \begin{pmatrix} i = 1, \ldots, m; \\ j = 1, \ldots, n; \\ k = 1, \ldots, r \end{pmatrix}$$

$$\rho = (1 + \frac{1}{mr\zeta})^{-\frac{1}{2}}, \quad \zeta = \sigma_T^2/(r\sigma_S^2 + \sigma_Z^2) \ .$$

(iii) Two-way nested model:

$$X_{ijk} = \mu + T_i + R_{ij} + Z_{ijk} \quad \begin{array}{l}(i = 1, \ldots, n; \\ j = 1, \ldots, m; \\ k = 1, \ldots, r)\end{array}$$

$$\rho = (1 + \frac{1}{mr\zeta})^{-\frac{1}{2}} , \quad \zeta = \sigma_T^2/(r\sigma_R^2 + \sigma_Z^2) \ .$$

These two-way models can be extended to n-way models since we are interested only in choosing T_i .

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